Diplomarbeit – Diploma Thesis

Dirac Fermions in $2+1$ Dimensions with Random Mass Distribution

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Preface

The random bond Ising model (RBI) was proposed to include disorder into the standard Ising model which was already understood quite well. It prooved to be harder to tackle than expected. Although some numerical results are available, there are still a lot of open questions.

One of the most valuable tools when dealing with disordered and impure systems is super-symmetry. The application of super-symmetric methods in condensed matter theory was initiated by Efetov in 1983. Recent developments of integration theory in super-spaces made it possible to express integrals defined over some classical Lie group in terms of integrals over superfields, that live on a coset space. These kind of identities are called "color-flavor transformations" since integrals similar to them arise in the context of quark and gluon fields.

The main goal of this thesis is to obtain a the lattice version of a non-linear sigma model for the RBI with binary probability distribution. Chapter 1 is intended to introduce the physical background of the Ising model, disorder treatment and the super-symmetric tools needed to do this. In chapter 2 a new color-flavor transformation is derived that relates an integral over the special-orthogonal group – SO(N) – to an integral over the symmetric super-coset-space Osp(2n|2n)/Gl(n|n). Chapter 3 is dedicated to the application of the color-flavor transformation derived in chapter 2. Here the random bond Ising model is mapped onto the lattice version of a sigma model using this transformation.

Contents

Abstract – Zusammenfassung

Das Ziel dieser Arbeit ist die Herleitung der Gitterversion eines nichtlinearen Sigmamodelles für das *Random-Bond*-Ising Modell (RBI) mit binärer Wahrscheinlichkeitsverteilung. [Cho 97] folgend wird das RBI auf ein Netzwerkmodell abgebildet. Dieses Netzwerkmodell hat große Ähnlichkeit mit dem 1988 von Chalker und Coddington vorgeschlagenen Modell für den ganzzahligen Quanten-Hall-Effekt [Chal 88]. Der Unterschied liegt im Auftreten des Zufalls: In dem mit dem RBI assoziierten Modell lebt ein $O(1) = \mathbb{Z}_2$ Zufall auf den Gitterpunkten *(nodes)* des Netzwerkmodelles, während im CC-Modell die Verbindungen *(links)* zwischen den Punkten einen U(1) Zufall tragen.

Die Abbildung des RBI-Hamiltonians führt über einen Dirac-Hamiltonian direkt zu dem bereits beschriebenen Netzwerkmodell. Um dem nichtlinearen Sigmamodell näher zu kommen, wird das Integral über die O(1) durch ein Integral über einen symmetrischen Superraum ersetzt. An diesem Punkt setzt eine neuartige Transformation an. Diese Transformation – die *Color-Flavor*-Transformation (C-F) – wird für den allgemeinen Fall SO(N) bzw. O(N) hergeleitet. Die Anwendung erfolgt dann auf das zum RBI gehörige Netzwerkmodell – ähnlich wie in [Zirn 97] für das CC-Modell. Die dabei auftauchenden Spezialfälle sind die O(1) und die SO(1). In diesem Zusammenhang ist bemerkenswert, daß die C-F-Transformation auch für kleine N noch exakt ist.

Für die resultierende Gitterwirkung besteht nun die Hoffnung, daß sich ein Kontinuums-Grenzwert bilden läßt, der dann zu einem nichtlinearen Sigmamodell führt. Aus diesem Sigmamodell sollten sich die kritischen Exponenten für das Skalenverhalten der Spin-Korrelationlänge im multikritischen Punkt des RBI entlang der Phasengrenze sowie entlang der sognannten "Nishimori-Linie" ermitteln lassen.

Chapter 1

Introduction

1.1 Physical Motivation

1.1.1 The History of the Ising Model

At the beginning of this century the physics of mesoscopic condensed matter systems – i.e. systems whose behavior is governed by quantum coherence over scales that are larger than the system size – was rather a minor field. Most tools – experimental as well as theoretical – necessary to investigate those systems were not at hand. The dawning of quantum physics during the first quarter of the century and the technological improvements thereby induced gave physicists the necessary equipment to enter the mesoscopic world.

From the very start of this development the community was interested in the description of the magnetic properties of the systems under consideration and – quite naturally – the impact of quantization on those properties. Since it was realized that the theories that would cover everyday experience were by far too complicated to start with, physicists were focusing on simple "toy" models. Those either ignored forces of interacting particles or did other cruel things to the underlying physics. But still these models were applicable to real physical systems and produced even at times results that complied better with experiment than any other theory before. The deeper reason for this is that for a wide range of physical systems some of the observed quantities do not depend on the underlying microscopical details, but only on universal symmetries.

One of these models was developed by E Ising in his 1925 PhD thesis [Isin 25] and is nowadays widely known as the "Ising model". It describes *Elementarmagnete*[1] on a square lattice that experience a coupling to their next neighbors – interaction with elementary magnets on all other sites are ignored[2]. Since a

[1] germ.: elementary magnets, nowadays identified with electron spins

[2] In fact, Ising gives already a qualitative argument for the physical justification for this procedure.

perfect lattice is assumed, the coupling constant is being kept constant over the whole lattice. The Hamiltonian of this system is then:

$$H = -J \sum_{\langle i,j \rangle} S_i S_j \tag{1.1}$$

where S_i denotes the spin operator for site i and summation is assumed to range over adjacent sites only.

This model was studied intensely in the subsequent years. Peierls proved in 1936 the existence of an ordered low-temperature phase [Peie 36] for $d = 2$. In 1941 Kramers and Wannier used transfer matrix and duality techniques to tackle the Ising model [Kram 41]. Finally, in 1944, Onsager was able to calculate the critical temperature and the free energy of the 2D Ising model thereby solving the most fundamental questions. In the sequel their methods were improved and various other physical quantities were derived, e.g. the spontaneous magnetization etc. In the 60s it was shown that – for dimension 4 and higher – a semi-classical mean field theory produced exact solutions for the Ising model. There was (and still is) no analytical solution to the 3D Ising models.

1.1.2 The Advent of Randomness

Realistic physical systems cannot be considered to be pure in any respect. This fundamental fact was usually ignored by the theories of condensed matter systems of the early days. Nevertheless the ever increasing precision of experimental techniques made it necessary to incorporate impurity into the theories.

1968 McCoy and Wu introduced an Ising model that included impurity induced disorder [McCo 68]. They allowed the coupling parameters between adjacent rows to fluctuate, i.e. the vertical paramters could take arbitrary values. At the same time the horizontal coupling parameters were kept constant.

In the following years the interest focused on the random bond Ising model (RBI), with random sign but constant modulus in the coupling parameter (binary distribution). The probability distribution was then:

$$P(J_{ij}) = p\,\delta(J_{ij} - J) + (1 - p)\,\delta(J_{ij} + J) \tag{1.2}$$

Here p is the probability to have positive J on bond $\langle i, j \rangle$. When we take a closer look at the phase diagram we see that for the pure Ising system we have a phase transistion from a ferromagnetic phase to a paramagnetic phase at finite temperature T_c. On the $T = 0$ axis we find a quantum critical point at which we have a spin-glass transition on the axis. Nishimori [Nish 81] was able to perform an exact calculation of various physical quantities for this RBI on a line in the phase diagram that was subsequently named after him. This Nishimori line is given by $\tanh \frac{J}{kT} = 2p - 1$. For the binary distribution (1.2) Nishimori arrives at $\langle\langle E \rangle\rangle = -N(N-1)(2p-1)/2$ as an expression of the internal energy. The

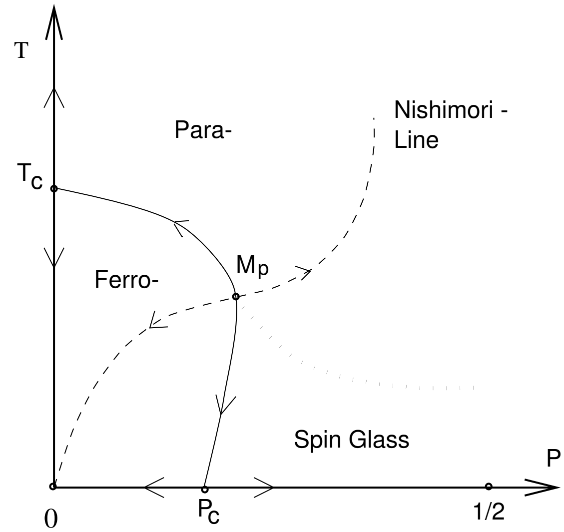

Figure 1.1: Phase diagram for the RBI

renormalization flow "above" the Nishimori line is directed towards the Ising critical point, while below the line it points towards the quantum critical point on the $T = 0$ axis. This indicates domination of the ferromagnetic viz. the spin-glass correlation. The intersection of the Nishimori line with the phase boundary is expected to be a multicritical point.

Further research on the random-bond Ising model was done by Dotsenko and Dotsenko [Dots 83], Shankar [Shan 87] and most recently by Singh and Adler [Sing 96] and Cho and Fisher [Cho 97]. Binder and Young were able to give a strong argument against a finite temperature spin-glass phase for the two-dimensional model [Bind 86].

The (approximate) p-T phase diagram representing our current knowledge on the RBI is given in figure 1.1. While we know a lot of the properties of the Nishimori multicritical point, the critical exponents are still only vaguely known. The major goal of this thesis is to take a further step towards a sigma model for the RBI.

1.1.3 QHE and Network Models

When mapping the RBI onto a non-linear sigma model we will make use of a network model. These type of models were developed in the framework of the quantum Hall effect (QHE). To give some background we give a very short review of their history.

In 1980 v. Klitzing et al. discovered some unusual behavior in 2D systems: the Hall conductances σ_{xx} and σ_{xy} were no longer linear in the applied magnetic

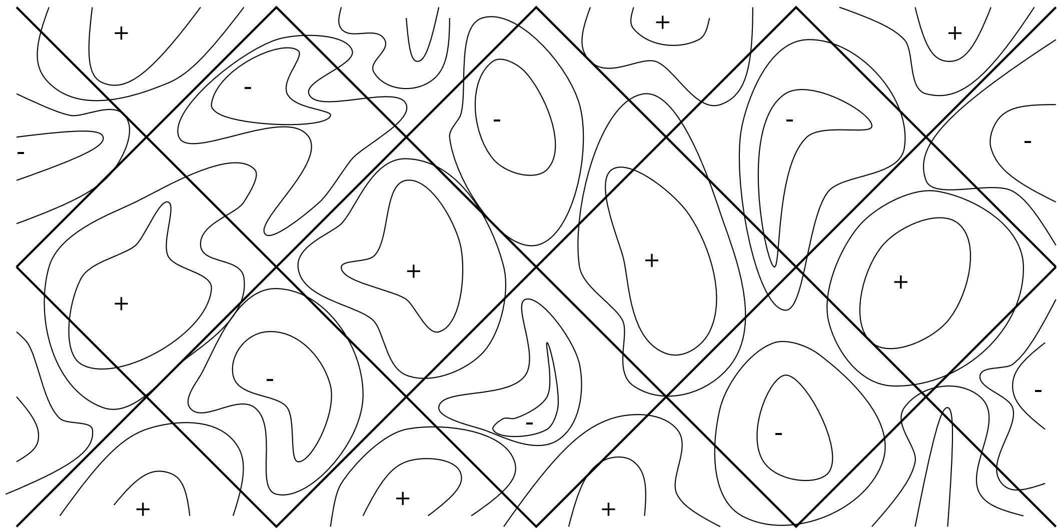

Figure 1.2: The Chalker-Coddington (CC) network model

field but seemed to be quantized. This phenomenon – the quantum Hall effect – was easy to reproduce, but theorists had a hard time understanding the physics of this effect. Among the first qualitative phenomenological ideas were Landau levels with extended states only at the center of each level; later field theories with topological terms followed.

A very remarkable ansatz to the problems of the QHE with disorder was proposed by Chalker and Coddington [Chal 88]. The two-dimensional conducting (square) lattice is modeled in the following way: Areas of higher and lower potential – that are induced by disorder – form "hills" and "wells". If the energy of the electrons is still low they are confined to the wells and their "guiding center" moves on a deformed circle. When raising the energy to the point where the wells are completely filled they touch (i.e. they come closer than the magnetic length l_c) and the electrons start to tunnel to adjacent wells thus percolating through the system. This behavior is caricaturized by a network of squares whose boundaries represent equipotential lines. A single line is usually called a link. On the links the electrons can move in one direction. At the end of each link the electron has to pass a node where it either continues to move along its current equipotential line or it tunnels through to some adjacent equipotential (cf. figure 1.2).

The wavefunctions are now represented by complex numbers living on the links. Randomness is incorporated in the following way: Each electron acquires an Aharonov-Bohm phase when passing a certain link. The amount of phase thus gained is proportional to the length of the link under consideration. When associating *random* phases with the links, and thus random lengths, we can quite naturally incorporate disorder into our model.

Although this model simplifies the real system to a large extend, still it is very hard to solve analytically. After some experience was gained about how to map the CC model on a corresponding Dirac Hamiltonian or how to treat it in terms of spin chains, the latest developments made it possible to map the problem on a non-linear sigma model [Zirn 97].

In chapter 3 we will review the mapping of the RBI onto a CC-like network model (cf. [Cho 97]) which in turn can be represented by a non-linear sigma model. Randomness occurs in this effective network model not on the links but on the nodes in terms of the sign of the coupling parameter. Dealing with the randomness requires us to use the color-flavor transformation, which is derived in chapter 2.

1.2 Mathematical Background

The key for the derivation of the sigma-model is the application of an integral identity – the color-flavor transformation – between an integral defined on the group manifold of the special orthogonal group in N dimensions – the SO(N) – and an integral defined over the symmetric super-space of type CI|DIII[3]. A self-contained introduction to integration theory over symmetric super-manifolds is certainly beyond the scope of this thesis – a good starting point for this is [Bere 87] and [Zirn 96b]. Furthermore we will make extensive use of the theory of Lie groups and Lie algebras which is also not introduced here as well. For a sound introduction to this field see [Helg 78]. We will look at some key definitions and theorems of theory of Lie groups and super manifolds only as a means of introducing our notation.

1.2.1 Lie Groups

Recall some definitions:

Definition 1.1 *1. A* Lie group *is a group G which is also an analytic manifold such that the mapping $(\sigma, \tau) \mapsto \sigma\tau^{-1}$, $\sigma, \tau \in G$ of the product manifold $G \times G$ into G is analytic.*

2. Let g be a vector space over a field K with characteristic 0. g is called Lie algebra *if there exists a bilinear mapping $[\bullet, \bullet]$ from $g \times g$ into g with the following properties:*

(a) $[X, Y] = -[Y, X]$

(b) $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$

[3]"CI" and "DIII" are two of Cartans symmetric spaces (cf. [Helg 78]); the classification for some of the symmetric super-spaces is given in [Zirn 96b].

To make a connection between Lie groups and algebras we have to introduce the *exponential mapping.*

Definition 1.2 *The mapping given by*

$$\exp : X \mapsto \exp X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

is called the exponential mapping. *$X \in g$ – a Lie algebra – is supposed to be given in some matrix representation.*

It can now be shown that with every Lie algebra comes at least one Lie group corresponding to it. The classification of the Lie algebras and groups was performed at the beginning of this century, mainly by E. Cartan. Again, this is reviewed in [Helg 78].

In order to integrate in the space of continous functions $\mathcal{C}(G)$ over the group manifold G we need a measure. If G is compact or $\mathcal{C}$ is restricted to the continous functions with compact support, we are guaranteed to have a so-called "Haar-measure" (cf. [Rich 81]) which is left-invariant, i.e. $\int_G f(hg)dg = \int_G f(g)dg$.

1.2.2 Generalized Coherent States

In the proof of the central theorem of this thesis we will make extensive use of so-called "generalized coherent states". Again we will introduce here only the basic concepts of these objects. For a general introduction we refer to [Pere 86].

Let G be an arbitrary Lie group and T_g its unitary irreducible representation acting on the Hilbert space $\mathcal{H}$. Choose some state $|0\rangle$ in that Hilbert space and consider then states $|g\rangle = T_g|0\rangle$. Take H to be the maximal subgroup of G, whose elements satisfy $T_h|0\rangle = |0\rangle \exp(i\alpha(h))$ (α denotes here some complex valued function of h). H is then referred to as the "isotropy subgroup" of G.

Definition 1.3 *The system of states $\{|g\rangle : |g\rangle = T_g|0\rangle\}$, where g are elements of the Lie group G, T is an unitary irreducible representation of G acting on some Hilbert space $\mathcal{H}$ and $|0\rangle$ is a fixed vector of that space, is called the coherent state system $\{T, |0\rangle\}$.*

Let H be the isotropy subgroup for T, i.e. the maximal subgroup that leaves an generalized coherent state unchanged besides an arbitrary factor. Then a coherent state $|g\rangle$ with $g \in G$ is determined by a point $x = \pi(g) \in G/H$ – the coset space of G and H – by $|g\rangle = \exp(i\alpha(h))\,|\pi(g)\rangle$. π is here the canonical mapping from the fiber bundle G onto its base G/H.

Among the most important properties of the coherent states we have the fact that they resolve the identity operator due to the irreducibility of the representation of G on $\mathcal{H}$. This can be seen as follows:

On every Lie group G there exists a left invariant measure dg, called the "Haar measure" (s.a.). It induces a measure dx on the (homogeneous) coset space $X = G/H$. Let us consider the operator

$$B = \int dx|x\rangle\langle x| \quad x \in X = G/H$$

assuming convergence. It is quite easy to see that $T_g B T_g^{-1} = B$, i.e. that B commutes with every T_g and must therefore be a multiple of the identity operator according to Schur's lemma. We can write then $B = c \cdot I$. To fix c calculate the expectation value with an arbitrary normalized [4] state $|g\rangle$:

$$\langle g|B|g\rangle = \int |\langle g|x\rangle|^2 dx = \int |\langle 0|x\rangle|^2 dx = d$$

To arrive at the promised resolution of unity we have to adjust our measure dx by introducing a new factor d^{-1}: $d\mu(x) = dx \cdot d^{-1}$. With this measure we have:

$$I = \int d\mu(x)|x\rangle\langle x| \tag{1.3}$$

1.2.3 Super-Analysis

A good introduction to super-analysis is given by [Efet 83] and [Bere 87]. Here we will recall some basics on Grassmanian (also known as fermionic) variables and the generalization of standard differential geometry to super-spaces. We start with the definition of Grassmann numbers:

Definition 1.4 *Let ξ_n be n distinct objects obeying the anti-commutation rule:*

$$\{\xi_i, \xi_j\} := \xi_i\xi_j + \xi_j\xi_i = 0$$

These objects are the (free) generators of the Grassmann algebra Λ.

Remarks:

1. It is quite important to note that the square of a generator vanishes. The elements of Λ can then be written as:

$$\eta = \sum_\pi f^{i_1 \cdots i_n} \xi^{i_{\pi(1)}} \ldots \xi^{i_{\pi(n)}}$$

   The sum is understood to run over all permutations of the indices, whereas the f are c-numbers. Taylor expansion stops therefore at finite order – for one generator this means that the series has a cut-off even at first order.

[4] i.e. $\langle g|g\rangle = 1$

2. Elements with an even number of Grassmann generators do actually commute. We can now define a parity for those elements of the algebra that have either an even or an odd number of generators in the following way: To those numbers that are composed of commuting elements only we assign parity 0, whereas the purely anti-commuting elements are assigned parity 1. This induces canonically a $\mathbb{Z}_2$ grading. Parity p elements span the space Λ_p in such a way that we have for the whole Grassmann algebra:

$$\Lambda = \Lambda_0 + \Lambda_1$$

3. The differentiation is defined just like ordinary differentiation with the exception that it is possible to define left and right derivatives due to the anti-commuting property of the Grassmann numbers.

4. We can also define integration over the Grassmann variables. The two types of integrals possible are then defined to be:

$$\int d\xi_i = 0 \qquad \int d\xi_i \, \xi_j = \delta_{ij}$$

A closer examination of the defining equations shows that integration and differentiation are just the same. We therefore get:

$$\int \bigotimes_i d\xi_i f(\xi_1, \ldots, \xi_n) = f(1, 1, \ldots, 1)$$

For a change in the integration variables we get from the above relations:

$$\int d\xi f(\xi) = \det A^{-1} \int d\eta f(A\eta), \text{ for } \xi = A\eta$$

and

$$\int d\xi f(\xi) = \int d\xi f(\xi + \eta).$$

One of the reasons to consider integration of Grassmann variables is the frequent occurrence of Gaussian integrals over Grassmann and super-vectors. Those integrals can be transformed – just like in the case of commuting variables – into determinants. We have

- for commuting ("bosonic") variables:

$$\int \bigwedge_i dz^i \wedge d\bar{z}^i \exp\left(-\bar{z}_i A_{ij} z_j\right) = \det{}^{-1} \mathbf{A}$$

- for anti-commuting ("fermionic") variables:

$$\int \bigotimes_i d\xi_i \otimes d\bar{\xi}_i \exp\left(-\bar{\xi}_i A_{ij} \xi_j\right) = \det \mathbf{A}$$

More important, arbitrary matrix elements can be expressed in terms of super-integrals. This is done by application of the above formulas. Thus we get:

$$\int D\psi\, D\bar{\psi}\, \psi_{i,\sigma} \bar{\psi}_{j,\sigma'} \exp\left(-\bar{\psi}_I A_{IJ} \psi_J\right) = A_{(i,\sigma)(j,\sigma')}\, \mathrm{SDetA}$$

An even more general formula for the expression of matrix elements by super-Gauss integrals is proved in appendix D.

It is now possible to consider "super-manifolds". These manifolds are defined to be locally the tensor product of the algebra of analytic functions over an open subset $U \subset \mathbb{R}^n$ with some Grassmann algebra Λ. This is described in [Bere 87]. It turns out that some of these super-manifolds – the Riemannian symmetric ones – correspond to physical universality classes. They are examined in the light of their physical relevance in a recent paper [Zirn 96b].

The super-manifold under consideration is a coset space of the *ortho-symplectic* super-group Osp(2n|2n) and the *general linear* super-group Gl(n|n). These super-groups and their corresponding group manifolds can be represented by $(2n+2n) \times (2n+2n)$ viz. $(n+n) \times (n+n)$ super-matrices. Consider for example the Osp representation: Take $4n \times 4n$ matrices and block-decompose them into $2n \times 2n$ blocks: $g = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix}$. This grading is taken to be in bosons and fermions, i.e. the g_{00} and g_{11} correspond to the BB and FF block respectively. They have therefore c-numbers as elements. The off-diagonal blocks have Grassmann numbers as entries.

During the proof of the color-flavor transformation in chapter 2 we will perform a "second quantization" and thus encounter super-Fock spaces. They can be thought of as the usual Fock spaces, but this time bosons as well as fermions can be created or destroyed. The super-creators and super-annihilators that act on the super-Fock space are labeled $\bar{c}^i_A$ viz. c^i_A where i is some "color" index and A is a "flavor" index, discriminating bosons and fermions. The reason for this naming scheme as well as the range of the indices will become clear during the proof.

Chapter 2

Color-Flavor Transformation

In this chapter we will prove the following remarkable identity:

Theorem 2.1 *The following equation holds (using the quantities defined below):*

$$
\begin{aligned}
&\int_{\mathrm{SO(N)}} dO \, \exp\left(\bar{\psi}^i_A O^{ij} \psi^j_A\right) = \int D\mu_N(Z,\tilde{Z}) \, \exp\left(\bar{\psi}^i_A Z_{AB} \bar{\psi}^i_B + \psi^i_A \tilde{Z}_{AB} \psi^i_B\right) + \\
&\quad + \int D\mu_N(Z,\tilde{Z}) \exp\left(\bar{\psi}^i_{\hat{A}} Z_{\hat{A}\hat{B}} \bar{\psi}^i_{\hat{B}} + \bar{\psi}^i_{\hat{A}} Z_{\hat{A}(e,F)} \psi^i_{(e,F)} + \psi^i_{(e.F)} Z_{(e,F)\hat{B}} \bar{\psi}^i_{\hat{B}} + \right. \\
&\quad \left. + \psi^i_{\hat{A}} \tilde{Z}_{\hat{A}\hat{B}} \psi^i_{\hat{B}} + \psi^i_{\hat{A}} \tilde{Z}_{\hat{A}(e,F)} \bar{\psi}^i_{(e,F)} + \bar{\psi}^i_{(e,F)} \tilde{Z}_{(e,F)\hat{B}} \psi^i_{\hat{B}}\right)
\end{aligned}
\tag{2.1}
$$

This equation relates an integral which is defined over the special orthogonal group in N dimensions to a sum of two integrals which are defined over a symmetric super-space of class (CI|DIII). The upper index i – also sometimes referred to as "color" index[1] – has range $1, \ldots, N$. The lower – or "flavor" – index A is a multi-index (a, σ) with $a = 1, \ldots, n$ and σ takes values in $\{B, F\}$. Here n is arbitrary, but usually dictated by the physical application. Flavor indices $\hat{A}$ with a hat are supposed to run over the whole range but (e, F). The reason for excluding this very entry and performing a creator/annihilator exchange is given in the subsequent derivation of the theorem. The ψ s and $\bar{\psi}$ s are super-variables, i.e. can be written in terms of c-numbers and Grassmann numbers, depending on the value of σ in the flavor index. While O are simply elements of the SO(N) and dO is the usual left invariant Haar measure on the Lie group, the Z and $\tilde{Z}$ super-matrices parameterize the coset space (CI|DIII). $D\mu_N$ is the normalized measure on the coset space, given by:

$$D\mu_N(Z,\tilde{Z}) = D(Z,\tilde{Z}) \; \mathrm{SDet}(1 - \tilde{Z}Z)^{-N}$$

[1]Integrals similar to the left-hand side appear sometimes in lattice-gauge theory. There, ψ corresponds to quarks, while the O s are similar to color-carrying gluons.

where $D(Z, \tilde{Z})$ denotes the "flat" Berezin measure[2].

2.1 Preliminaries

In order to prove identity (2.1) we consider two vector spaces V and W – $\mathbb{C}^N$ and $\mathbb{C}^{2n|2n}$ respectively. These spaces will be referred to as physical or color space viz. auxiliary or flavor space. The auxiliary space is a $\mathbb{Z}_2$ graded sum of a bosonic space $W_B = \mathbb{C}^{2n}$ and a fermionic space $W_F = \mathbb{C}^{2n}$, i.e. $W = W_B \oplus W_F$. We will later utilize the Gaussian Berezin integral over the symmetric Riemannian super-space $\mathrm{Hom}_\lambda(V, W) \times \mathrm{Hom}_{\tilde{\lambda}}(W, V)$ where $\mathrm{Hom}_\lambda(W, V) \overset{\text{def}}{=} \lambda_0 \otimes \mathrm{Hom}(W_B, V) + \lambda_1 \otimes \mathrm{Hom}(W_F, V)$ with $\lambda = \lambda_0 + \lambda_1$ the Grassmann algebra with $N = \dim_{\mathbb{C}}(\mathrm{Hom}(W_F,V))$ generators.

Since our final goal is to integrate over the SO(N) and transform this into an integral over the Osp(2n|2n)/Gl(n|n) coset space, we have to relate them in some way. We will do this by relating their corresponding algebras.

If we now consider elements $\psi, \tilde{\psi}$ of the spaces of homomorphisms from W to V viz. V to W, we will use them to construct endomorphisms in W and V by simple concatenation. To force the proper group structure on these endomorphisms we have to put constraints on their form. In matrix representation this is usually done by requiring the algebra elements to fulfill some condition like

$$\mathcal{A}\mathcal{J} + \mathcal{J}\,\mathcal{A}^T = 0 \;\Leftrightarrow\; \mathcal{A} = -\mathcal{J}\,\mathcal{A}^T\,\mathcal{J}^{-1} \tag{2.2}$$

for all elements $\mathcal{A}$ of the algebra under consideration. In our case the $(\cdot)^T$ denotes the usual matrix transposition or super transposition for the so(N) viz. the osp(2n,2n). Using now our homomorphisms we can model the algebras they belong to by imposing proper constraints on them:

$$\begin{aligned} (\tilde{\psi}\psi) &= -\gamma(\tilde{\psi}\psi)^T\gamma^{-1} \in \mathrm{End}(W) \\ (\psi\tilde{\psi}) &= -\mathcal{C}(\psi\tilde{\psi})^T\mathcal{C}^{-1} \in \mathrm{End}(V) \end{aligned} \tag{2.3}$$

To force $\mathrm{End}(V)$ to be an so(N) algebra we simply set $\mathcal{C} = \mathbb{1}_N$. But this already puts a constraint on γ, since:

$$\begin{aligned} \tilde{\psi}\psi &= -\gamma\psi^T\tilde{\psi}^T\gamma^{-1} \\ &= -\left(\gamma\psi^T\mathcal{C}^{-1}\right)\left(\mathcal{C}\tilde{\psi}^T\gamma^{-1}\right) \end{aligned} \qquad \begin{aligned} \psi\tilde{\psi} &= -\mathcal{C}\tilde{\psi}^T\psi^T\mathcal{C}^{-1} \\ &= -\left(\mathcal{C}\tilde{\psi}^T\gamma^{-1}\right)\left(\gamma\psi^T\mathcal{C}^{-1}\right) \end{aligned}$$

A proper choice for ψ and $\tilde{\psi}$ is then:

[2] For further details see [Zirn 96b] and [Roth 87]. It is important to note that this relation is only locally valid. When defining a global measure, so-called "anomalies" (aka. boundary terms) arise which one has to take into account.

$$\tilde{\psi} = -\gamma\psi^T\mathcal{C}^{-1} \qquad \psi = \mathcal{C}\tilde{\psi}^T\gamma^{-1} \tag{2.4}$$

We proceed by using this relation to get some condition for γ:

$$\begin{aligned}
\psi &= \mathcal{C}\tilde{\psi}^T\gamma^{-1} \\
&= -\mathcal{C}\left(\gamma\psi^T\mathcal{C}^{-1}\right)^T\gamma^{-1} \\
&= -\underbrace{\mathcal{C}(\mathcal{C}^{-1})^T}_{=\mathbb{1}\ \text{, since } \mathcal{C}=\mathbb{1}}\psi^{TT}\gamma^T\gamma^{-1} \\
&= -\psi\sigma\gamma^T\gamma^{-1}
\end{aligned}$$

Thus we arrive at:

$$\gamma = -\gamma^T\sigma \tag{2.5}$$

As already noted above σ is the superparity $\mathrm{diag}(\mathbb{1}_{2n}, -\mathbb{1}_{2n})$. With this constraint in mind we can choose γ to be:

$$\gamma = \begin{pmatrix} 0 & \mathbb{1}_n & 0 & 0 \\ -\mathbb{1}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_n \\ 0 & 0 & \mathbb{1}_n & 0 \end{pmatrix}$$

In [Bere 87] the osp(2n,2n) is defined to fulfill (2.2) with this γ (as $\mathcal{J}$)[3].

The two algebras occuring – so(N) and osp(2n,2n) – can now be represented in second quantized form by creator/annihilators pairs (cf. lemma A.1):

Definition 2.1 *1. The generators of the* osp(2n,2n) *can be written as*

$$\begin{array}{llll}
B^{ab} = \bar{b}^i_a\bar{b}^i_b & F^{ab} = \bar{f}^i_a\bar{f}^i_b & G^{ab} = \bar{b}^i_a\bar{f}^i_b & \\
B^b_a = \frac{1}{2}\left(b^i_a\bar{b}^i_b + \bar{b}^i_b b^i_a\right) & F^b_a = \frac{1}{2}\left(f^i_a\bar{f}^i_b - \bar{f}^i_b f^i_a\right) & G^b_a = \bar{b}^i_a f^i_b & \tilde{G}^b_a = b^i_a\bar{f}^i_b \\
B_{ab} = b^i_a b^i_b & F_{ab} = f^i_a f^i_b & G_{ab} = b^i_a f^i_b &
\end{array}$$

Sometimes they will be referred to as color singlet or flavor operators $\hat{C}_{ab}$.

2. *The generators of the* so(N) *algebra can be represented by*

$$\hat{F}^{ij} = \bar{b}^i_a b^j_a - b^i_a\bar{b}^j_a + \bar{f}^i_a f^j_a + f^i_a\bar{f}^j_a$$

They will be referrred to as flavor singlet or color operators.

[3] In fact, Berezin chose $\mathcal{J}$ to have simply $\mathbb{1}_{2n}$ in the FF sector instead of $\sigma_x \otimes \mathbb{1}_n$. A simple calculation shows that nothing is changed, as can also be seen from the commutators of the osp(2n,2n) generators in the FF sector.

2.2 The osp(2n,2n) and the Flavor Sector

The principle idea in the following proof of the color-flavor transformation is to show the equality of two different projectors on the flavor sector[4]. The first way to describe the flavor sector is given by the very definition of it: Being the part of the super-Fock space that gets annihilated by the flavor singlet operators $\hat{F}^{ij}$. On the other hand we will show that the flavor sector is completely covered by acting with the color singlet operators, which form an osp(2n,2n) algebra, on the vacuum state $|0\rangle$ and on the Fermi-baryonic[5] state $\bar{F}^B_{(a)}|0\rangle$. The baryonic state is defined to have one fermion of every color, but all with the same flavor[6]:

$$\bar{F}^B_{(a)}|0\rangle := \bar{f}^1_a \dots \bar{f}^N_a |0\rangle$$

A crucial difference to [Zirn 96a] is that the flavor sector decomposes into two parts – the vacuum and the baryonic subsector – each of which cannot be reached by a multiple action of the osp(2n,2n) onto their states. This was already shown for the large N limit in [Zirn 96b] by using the saddle-point approximation and thus anticipated to hold also for arbitrary N. The deeper reason behind this is the existence of another invariant tensor for the SO(N), namely the total antisymmetric tensor in N dimensions: $\epsilon^{i_1 \dots i_N}$.

The form of the generators of the so(N) and the osp(2n,2n) respectively is derived in the appendix.

To show that osp(2n,2n) acts irreducibly on each of the two subsectors of the flavor sector we will take three steps: First we will show that the vacuum and the baryonic state are elements of the flavor sector. Then we are going to show that the color singlet operators (i.e. the osp generators) and the flavor singlet operators commute. This shows that at least all osp(2n,2n) generated states are in fact color neutral. The last thing to prove is the non-existence of further states. This will be achieved by using the tensor-space representation of the so(N), limiting the flavor states (defined through the flavor singlet operator) to those that can be reached by multiple osp(2n,2n) action onto the vacuum and the baryonic state. It is the last step, where the additional invariant tensor enters the game. Obviously, when switching from the SO(N) to the O(N) this tensor no longer occurs.

2.2.1 The Flavor Sector

We start with a definition of some ingredients that will play a crucial rôle in our investigation:

[4] The flavor sector is a subspace of the super-Fock space. It will be defined immediately.

[5] We use the term "baryonic" since every flavor occurs in the state under consideration, just like every color occurs in the case of the constituting quarks in the hadrons.

[6] It will be shown that the flavor is of no importance.

Definition 2.2 *Let us consider a super-Fock space with vacuum $|0\rangle$, selected by*[7] $c^i_A|0\rangle = 0$. *The set of elements of the Fock space that vanishes under action of the so(N) generator – the flavor singlet operator $\hat{F}^{ij}$ – is called the* color neutral *or* flavor sector. *Elements of this sector will sometimes be denoted as $|\text{flavor}\rangle$.*

Remark: The flavor sector is actually the key to the whole proof given here. We will construct a projector on that very sector in two different ways and show that these projectors are identical.

The first thing to prove is that the vacuum $|0\rangle$ and the baryonic state $\bar{F}^B|0\rangle$ actually get killed by the flavor singlet operators:

Lemma 2.1 *The vacuum state $|0\rangle$ and the Fermi-baryonic state $\bar{F}^B_{(a)}|0\rangle$ are elements of the flavor sector:*

$$\hat{F}^{ij}|0\rangle = 0 \textit{ and } \hat{F}^{ij}\bar{F}^B_{(a)}|0\rangle = 0 \textit{ for all } i,j = 1 \dots N$$

Proof: When acting with $\hat{F}^{ij}$ onto $|0\rangle$ we get:

$$\begin{aligned} \hat{F}^{ij}|0\rangle &= \sum_a \bar{b}^i_a b^j_a - b^i_a \bar{b}^j_a + \bar{f}^i_a f^j_a + f^i_a \bar{f}^j_a |0\rangle \\ &= -n \cdot \delta^{ij} + n \cdot \delta^{ij} \\ &= 0 \end{aligned}$$

The second equality sign stems from the fact that for $i \neq j$ the creators and annihilators trivially commute viz. anti-commute, whereas for $i = j$ the constants arising from the exchange of the Bose and Fermi operators cancel exactly. For the Fermi-baryonic state we have:

$$\begin{aligned} \hat{F}^{ij}\bar{F}^B_{(b)}|0\rangle &= \sum_a \left(\bar{b}^i_a b^j_a - b^i_a \bar{b}^j_a + \bar{f}^i_a f^j_a + f^i_a \bar{f}^j_a\right)\left(\bar{f}^1_b \dots \bar{f}^N_b\right)|0\rangle \\ &= \left(0 - n \cdot \delta^{ij} + \delta^{ij} + (n-1) \cdot \delta^{ij}\right)|0\rangle \\ &= 0 \end{aligned}$$

Here we have applied the same idea as above. We still have to show that the flavor of the Fermi-baryonic state plays no rôle. This is established as soon as we can show that there is no problem in getting from $\bar{F}^B_{(a)}$ to $\bar{F}^B_{(b)}$ by a multiple action of the osp(2n,2n) generators:

[7] Here – as everywhere else – c^i_A denotes the annihilator for a particle of color i and flavor A, where A is a multi-index (a, σ), $a = 1, \dots, N$ and $\sigma \in \{B, F\}$. Naming and range of the indices is the same for the corresponding creator $\bar{c}^i_A$.

$$
\begin{aligned}
F_a^b \bar{F}^B_{(a)}|0\rangle &= \tfrac{1}{2}\left(f_a^i \bar{f}_b^i - \bar{f}_b^i f_a^i\right) \bar{f}_a^1 \ldots \bar{f}_a^N |0\rangle \\
&= \left(\bar{f}_b^1 \bar{f}_a^2 \ldots \bar{f}_a^N + \ldots + \bar{f}_a^1 \ldots \bar{f}_a^{N-1} \bar{f}_b^N\right)|0\rangle
\end{aligned}
$$

$$
\begin{aligned}
F_b^a F_b^a \bar{F}^B_{(a)} &= \left(\bar{f}_b^1 \bar{f}_b^2 \bar{f}_a^3 \ldots \bar{f}_a^N + \ldots + \bar{f}_b^1 \bar{f}_a^2 \ldots \bar{f}_b^N + \right.\\
&\quad + \bar{f}_b^1 \bar{f}_b^2 \ldots \bar{f}_a^N + \ldots + \bar{f}_a^1 \bar{f}_b^2 + \ldots \bar{f}_b^N + \\
&\quad + \ldots \\
&\quad \left. + \bar{f}_b^1 \bar{f}_a^2 \ldots \bar{f}_b^N + \ldots + \bar{f}_a^1 \ldots \bar{f}_b^{N-1} \bar{f}_b^N\right)|0\rangle
\end{aligned}
$$

Thus we arrive finally at:

$$
\bar{F}^B_{(b)} = \frac{1}{N!}\left(F_a^b\right)^N \bar{F}^B_{(a)}|0\rangle. \qquad \square
$$

In the following theorem we show by a simple argument that there is no way to arrive at the baryonic state by any multiple action of the osp(2n,2n) generators on the vacuum state and vice versa.

Theorem 2.2 *The flavor sector decomposes into at least two unconnected sets of states. In the one part lies the vacuum whereas the other contains the baryonic Fermi states. Neither of these parts can be reached by a multiple action of the* osp(2n,2n) *on an arbitrary state of the other.*

Proof: Each of the color singlet operators can create or destroy only pairs of particles with the same color or change the flavor of particles but not their color. □

To continue our program we have to prove the following lemma:

Lemma 2.2 *The generators of the* osp(2n,2n) *super-algebra commute with the flavor singlet operator, i.e.*

$$
\forall\, i,j \in \{1,\ldots,N\} \forall a,b \in \{1,\ldots,n\} \;:\; [\hat{F}^{ij}, \hat{C}_{ab}] = 0 \tag{2.6}
$$

Proof: To prove this relation we simply calculate the commutators:

$$
\begin{aligned}
[\hat{F}^{ij}, B^{ab}] &= \sum_{k,c} \bar{b}_c^i b_c^j \bar{b}_a^k \bar{b}_b^k - b_c^i \bar{b}_c^j \bar{b}_a^k \bar{b}_b^k - \bar{b}_a^k \bar{b}_b^k \bar{b}_c^i + \bar{b}_a^k \bar{b}_b^k b_c^i \bar{b}_c^j \\
&= \sum_{k,c} \delta_{ac}^{jk} \bar{b}_c^i \bar{b}_b^k + \delta_{bc}^{jk} \bar{b}_c^i \bar{b}_a^k - \delta_{ac}^{ik} \bar{b}_c^j \bar{b}_b^k - \delta_{bc}^{ik} \bar{b}_c^j \bar{b}_a^k \\
&= 0 \\
[\hat{F}^{ij}, B_{ab}] &= 0 \;\; \textit{(simile)} \\
[\hat{F}^{ij}, B_a^b] &= \tfrac{1}{2}\sum_{k,c} \bar{b}_c^i b_c^j b^k \bar{b}_b^k + \bar{b}_c^i b_c^j \bar{b}_b^k b_a^k - b_c^i \bar{b}_c^j b_a^k \bar{b}_b^k - b_c^i \bar{b}_c^j \bar{b}_b^k b_a^k \\
&\quad - b_a^k \bar{b}_b^k \bar{b}_c^i b_c^j - \bar{b}_b^k b_a^k \bar{b}_c^i b_c^j + b_a^k \bar{b}_b^k b_c^i \bar{b}_c^j + \bar{b}_b^k b_a^k b_c^i \bar{b}_c^j \\
&= \tfrac{1}{2}\sum_{k,c} -\delta_{ac}^{ik} b_c^j \bar{b}_b^k + \delta_{bc}^{jk} b_a^k \bar{b}_c^i + \delta_{cb}^{jk} \bar{b}_c^i b_a^k - \delta_{ac}^{ik} \bar{b}_b^k b_c^j - \delta_{ac}^{jk} \bar{b}_b^k b_c^i \\
&\quad + \delta_{bc}^{ik} \bar{b}_c^j b_a^k - \delta_{ac}^{ik} \bar{b}_c^j b_a^k + \delta_{ac}^{jk} \bar{b}_b^k b_c^i \\
&= 0
\end{aligned}
$$

The calculations for the pure fermionic and the mixed generators are very similar and will thus be omitted. □

2.2.2 The Tensor Space Representation of the SO(N)

To deal with the definition of the flavor sector we will now use the tensor space representation of the SO(N). Thus we can characterize the color neutral state using invariant tensors of the SO(N). A general introduction to this method is given in [Tung 85]. We can summarize the ideas as follows:

The most arbitrary state of the Fock space can be described in terms of:

$$|\text{state}\rangle = \sum F^{i_1 i_2 \cdots i_k}_{A_1 A_2 \cdots A_k} \bar{c}^{i_1}_{A_1} \ldots \bar{c}^{i_k}_{A_k} |0\rangle$$

If $R \in \text{SO(N)}$ is now a rotation in color space, such a state transforms as a number of copies of the vector representation $\bar{c}^i_A \mapsto \sum R^{ij} \bar{c}^j_A$. If $|\text{state}\rangle$ is now supposed to be color-neutral, then $\hat{F}|\text{state}\rangle = 0$ has to hold by definition. But this means that $F^{i_1 i_2 \cdots i_k}_{A_1 A_2 \cdots A_k}$ has to be composed of SO(N) invariant tensors in the color indices only.

For the SO(N) there are two relevant types of invariant tensors[8] (cf. [Tung 85]):

1. The tensor ξ^{ij} given by the metric being kept invariant by the SO(N). Since we are dealing with a positive definite metric this tensor can be written in matrix form by $\xi = \mathbb{1}_N$.

2. The total anti-symmetric tensor of rank N, $\epsilon^{i_1 \cdots i_N}$. This tensor guarantees that the elements of the SO(N) preserve the orientation, i.e. that the determinant of the orthogonal transformation is $+1$, as required by the virtue of it being a "special" group.

[8] Another invariant tensor is – quite naturally – the δ tensor. But this tensor is only of minor importance; its rôle for the Gl(N) is here being played by ξ. For a detailed discussion compare [Tung 85], ch. 13.

2.2.3 A Simple Example

Let us now see what happens in a low-dimensional example:

Example: The so(2) *and the* osp(2,2)

The osp(2,2) generators are summarized in this table:

$$
\begin{array}{lll}
B^{11} = \sum_{i=1}^{2} \bar{b}^i \bar{b}^i & F^{11} = 0 & G^{11} \sum_{i=1}^{2} \bar{b}^i \bar{f}^i \\
B^1_1 = \frac{1}{2} \sum_{i=1}^{2} b^i \bar{b}^i + \bar{b}^i b^i & F^1_1 = \frac{1}{2} \sum_{i=1}^{2} f^i \bar{f}^i - \bar{f}^i f^i & G^1_1 = \sum_{i=1}^{2} \bar{b}^i f^i \\
B_{11} = \sum_{i=1}^{2} b^i b^i & F_{11} = 0 & \tilde{G}^1_1 = \sum_{i=1}^{2} b^i \bar{f}^i \\
 & & G_{11} = \sum_{i=1}^{2} b^i f^i
\end{array}
$$

The flavor states are:

$$|\text{flavor}\rangle = \sum_{\pi} f_{\sigma_1 \ldots \sigma_{2k}}(\pi) \bar{c}^{i_1}_{\sigma_1} \ldots \bar{c}^{i_{2k}}_{\sigma_{2k}} \, \mu^{i_{\pi(1)} i_{\pi(2)}} \ldots \mu^{i_{\pi(2k-1)} i_{\pi(2k)}}$$

where $\pi \in S_{2k}$, $\mu \in \{\xi, \epsilon\}$ and $f_{\sigma_1 \ldots \sigma_{2k}}(\pi)$ is an arbitrary factor. If we now have states with more than one anti-symmetric tensor in the representation above, something very interesting happens: Each pair of ϵ-tensors can be represented in terms of some ξ-tensors:

$$\epsilon^{ij} \epsilon^{kl} = \xi^{ik} \xi^{jl} - \xi^{il} \xi^{jk} \tag{2.7}$$

This equality is obtained by proper contraction from the well known fact that the product of a full antisymmetric co- and a full antisymmetric contravariant tensor can be expressed in terms of the unit tensor δ. By this fact we can restrict ourselves to the case with none or one anti-symmetric tensor being present in the representation.

Applying this to our states we arrive at the following 2-particle states in the color neutral sector:

$$
\begin{array}{ll}
\bar{b}^1 \bar{b}^1 + \bar{b}^2 \bar{b}^2 |0\rangle & \frac{1}{2}\left(\bar{b}^1 \bar{b}^2 - \bar{b}^2 \bar{b}^1\right) |0\rangle = 0 \\
\bar{b}^1 \bar{f}^1 + \bar{b}^2 \bar{f}^2 |0\rangle & \frac{1}{2}\left(\bar{b}^1 \bar{f}^2 - \bar{b}^2 \bar{f}^1\right) |0\rangle \\
\bar{f}^1 \bar{f}^1 + \bar{f}^2 \bar{f}^2 |0\rangle = 0 & \frac{1}{2}\left(\bar{f}^1 \bar{f}^2 - \bar{f}^2 \bar{f}^1\right) = |0\rangle = \frac{1}{2} \bar{F}^B |0\rangle
\end{array} \tag{2.8}
$$

It is not difficult to see that the states on the right hand side are reachable from the baryonic state, whereas the states on the left hand side are generated by multiple action on the vacuum. There is no possibility to reach a state on the left from the right and vice versa by any osp action. When we consider higher-particle states we observe the same behavior.

This simple example illustrates the principal idea behind the tensor-space representation and gives us some idea of how to relate it to the algebraic representation of the flavor sector in higher dimensions.

2.2.4 The Irreducibility Theorem

The generalization of (2.7) to dimensions $N > 2$ is now crucial for the further proceedings. It is achieved by the following lemma:

Lemma 2.3 *The product of two anti-symmetric tensors can be written in terms of linear combinations of products of the metric:*

$$\epsilon^{i_1 \dots i_k} \epsilon^{j_1 \dots j_k} = \begin{vmatrix} \xi^{i_1 j_1} & \dots & \xi^{i_1 j_k} \\ \vdots & \ddots & \vdots \\ \xi^{i_k j_1} & \dots & \xi^{i_N j_k} \end{vmatrix} \tag{2.9}$$

Proof: By acting with ξ_{ij} k-times on $\epsilon^{\cdot\cdot}$ we get the covariant tensor $\epsilon_{\dots}$. We can then apply the fact (cf. [Sexl 76] for a proof) that

$$\epsilon^{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} = \delta^{i_1 \cdots i_N}_{j_1 \cdots j_N}$$

and use ξ^{ij} afterwards on $\delta^{i_1 \cdots i_N}_{j_1 \cdots j_N}$ to arrive at the statement above. □
Thus the final ingredient is no longer difficult to prove:

Lemma 2.4 *The elements of the flavor sector can be written in tensor space representation in the following form:*

$$\begin{aligned} |\text{flavor}\rangle &= \sum f_{A_1 \dots A_{2k}}(\pi) \xi^{i_{\pi(1)} i_{\pi(2)}} \dots \xi^{i_{\pi(2k-1)} i_{\pi(2k)}} \bar{c}^{i_1}_{A_1} \dots \bar{c}^{i_{2k}}_{A_{2k}} |0\rangle \\ &\mathit{or} \\ |\text{flavor}\rangle &= \sum \sum f_{A_1 \dots A_{2k+N}}(\pi) \xi^{i_{\pi(1)} i_{\pi(2)}} \dots \xi^{i_{\pi(2k-1)} \pi i_{\pi(2k)}} \epsilon^{i_{2k+1} \cdots i_{2k+N}} \times \\ &\qquad \times \bar{c}^{i_1}_{A_1} \dots \bar{c}^{i_{2k}}_{A_{2k}} \bar{c}^{i_{2k+1}}_{A_{2k+1}} \dots \bar{c}^{i_{2k+N}}_{A_{2k+N}} |0\rangle \end{aligned} \tag{2.10}$$

Remark: As soon as there are two or more bosons of same flavor among the generators that we contracted with the ϵ tensor, the whole expression vanishes, of course.

Proof: Since ξ and ϵ are the only relevant invariant tensors, all elements of the flavor sector (i.e. those elements of the Fock space that are invariant under SO(N) rotations) can be written in terms of the invariant tensors of this group, as indicated above. □

We finally arrive at the following theorem:

Theorem 2.3 (Irreducibility Theorem) *The generators of the* osp(2n,2n) *algebra act irreducibly on each of the disconnected subsectors of the flavor sector.*

Proof: This theorem can now easily be proven by collecting our lemmata and noting that

$$\sum \bar{c}^{i_1}_{A_1} \ldots \bar{c}^{i_k}_{A_k} \xi^{i_{\pi(1)} i_{\pi(2)}} \ldots \xi^{i_{\pi(k-1)} i_{\pi(k)}} |0\rangle \tag{2.11}$$

can be reached by multiple action of $\bar{c}^i_A \bar{c}^j_B \delta^{ij}$ (which is then B^{ab}, F^{ab} or G^{ab}, respectively) on the vacuum. Elements of the form

$$\sum \bar{c}^{i_1}_{A_1} \ldots \bar{c}^{i_k}_{A_k} \bar{c}^{i_{k+1}}_{A_{k+1}} \ldots \bar{c}^{i_{k+N}}_{A_{k+N}} \xi^{i_{\pi(1)} i_{\pi(2)}} \ldots \xi^{i_{\pi(k-1)} i_{\pi(k)}} \epsilon^{i_{k+1} \ldots i_{k+N}} |0\rangle \tag{2.12}$$

can now also be reached by multiple osp(2n,2n) action through:

1. If $\sigma_{k+1} = \ldots = \sigma_{k+N} = F$ then

   $$F^{a_{k+1}}_a \ldots F^{a_{k+N}}_a \bar{F}^B_{(a)} |0\rangle$$

   where $\bar{F}^B_{(a)}$ denotes the Fermi baryonic state with flavor a.

2. If (without loss of generality) $\sigma_{k+1} = B$ and $\sigma_{k+2} = \ldots = \sigma_{k+N} = F$ then

   $$G^{a_{k+1}}_a F^{a_{k+2}}_a \ldots F^{a_{k+N}}_a \bar{F}^B_{(a)} |0\rangle \tag{2.13}$$

   yields the right state.

3. For $\sigma_{k+1} = \sigma_{k+2} = B$ and $\sigma_{k+3} = \ldots = \sigma_{k+N} = F$ we have to look closer at the flavor of our bosons: If now $a_1 = a_2$ the state is killed. This can be proven by calculating the multiple action of $G^a_b G^a_b$. For $a_1 \neq a_2$ we arrive at:

   $$G^{a_{k+1}}_a G^{a_{k+2}}_a F^{a_{k+3}}_a \ldots F^{a_{k+N}}_a \bar{F}^B_{(a)} |0\rangle$$

   This result can be understood quite intuitively: Having more than one boson (i.e. commuting operator) of the same type coupled to the total anti-symmetric tensor means that the whole tensor must vanish.

This scheme can be trivially enlarged to a larger count of bosons. However it is important to stress that as soon as there are two or more bosons of the same flavor present the whole state vanishes. This reproduces precisely the behavior of the ϵ tensor. The creators not coupled to ϵ are taken care of by – as above – the metric tensor. □

Remark: Since every state of each of the parts of the flavor sector can be written as described in (2.11, 2.12, 2.13) it is easy to see that we can reach the vacuum $|0\rangle$ viz. the Fermi-baryonic state $\bar{F}^B_{(a)}|0\rangle$ quite naturally by acting with the corresponding "inverses" on these states, i.e. with B_{ab} on B^{ab}, B^a_b on B^b_a and so forth. Thereby we can reach every state of each of the two parts by a multiple action of color singlet operators on some "start" state.

2.3 Construction of the Projector

2.3.1 The Action of Osp(2n|2n)

In order to obtain our generalized coherent states we have to define an action of the Osp(2n|2n) on our Fock space. This action must actually be a Lie supergroup homomorphism. Since our flavor sector decomposes into two unconnected parts we have the freedom to choose different actions on them. To ease notation we will change the block structure of the matrices representing the elements of the group viz. algebra. While until now we had $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A representing the BB-sector etc., we will now order the elements so that A represents the $\bar{c}c$ sector, B the $\bar{c}\bar{c}$ sector, C the cc sector, and finally D the $c\bar{c}$ sector. We then define the action T_g^0 on the subsector containing the vacuum:

$$T_g^0 = \exp\left\{\bar{c}_A^i(\ln A)_{AB}c_B^i + \bar{c}_A^i(\ln B)_{AB}\bar{c}_B^i + c_A^i(\ln C)_{AB}c_B^i + c_A^i(\ln D)_{AB}\bar{c}_B^i\right\}$$

whereas the action T_g^B on the other subsector that contains the baryonic Fermi states is going to be:

$$\begin{aligned} T_g^B \;=\; \exp\Big\{ & \bar{c}_{\hat{A}}^i(\ln A)_{\hat{A}\hat{B}}c_{\hat{B}}^i + \bar{c}_{\hat{A}}^i(\ln A)_{\hat{A}(e,F)}\bar{c}_{(e,F)}^i \\ & + c_{(e,F)}^i(\ln A)_{(e,F)\hat{B}}c_{\hat{B}}^i + c_{(e,F)}^i(\ln A)_{(e,F)(e,F)}\bar{c}_{(e,F)}^i \\ & + \bar{c}_{\hat{A}\hat{B}}^i(\ln B)_{\hat{A}}\bar{c}_{\hat{B}}^i + \bar{c}_{\hat{A}}^i(\ln B)_{\hat{A}(e,F)}c_{(e,F)}^i \\ & + c_{(e,F)}^i(\ln B)_{(e,F)\hat{B}}\bar{c}_{\hat{B}}^i + c_{(e,F)}^i(\ln B)_{(e,F)(e,F)}c_{(e,F)}^i \\ & + c_{\hat{A}}^i(\ln C)_{\hat{A}\hat{B}}c_{\hat{B}}^i + c_{\hat{A}}^i(\ln C)_{\hat{A}(e,F)}\bar{c}_{(e,F)}^i \\ & + \bar{c}_{(e,F)}^i(\ln C)_{(e,F)\hat{B}}c_{\hat{B}}^i + \bar{c}_{(e,F)}^i(\ln C)_{(e,F)(e,F)}\bar{c}_{(e,F)}^i \\ & + c_{\hat{A}}^i(\ln D)_{\hat{A}\hat{B}}\bar{c}_{\hat{B}}^i + c_{\hat{A}}^i(\ln D)_{\hat{A}(e,F)}c_{(e,F)}^i \\ & + \bar{c}_{(e,F)}^i(\ln D)_{(e,F)\hat{B}}\bar{c}_{\hat{B}}^i + \bar{c}_{(e,F)}^i(\ln D)_{(e,F)(e,F)}c_{(e,F)}^i\Big\} \end{aligned}$$

The difference between T_g^0 and T_g^B lies in the rôle of the fermion annihilators of flavor e: In the vacuum subsector they are actually just the ordinary fermion annihilators while in the subsector containing $\bar{F}_{(e)}^B|0\rangle$ they act like hole-creators and are thus treated on an equal footing with the other fermion creators $\bar{f}_{\hat{a}}^i$. $\hat{A}$ viz. $\hat{a}$ denotes here the flavor indices that cover the complete flavor range except those fermions viz. particles with flavor e.

The well-definedness of the logarithm of a general element of the Gl(n|n) was shown in [Zirn 96a]. The same argument applies here *mutatis mutandi*.

Lemma 2.5 *The mapping $g \mapsto T_g$ is on both subsectors a super-group homomorphism.*

Proof: From [Helg 78] we learn that – for the differential of the exponential mapping from any Lie algebra to the corresponding Lie group – we have:

$$\begin{aligned}
\mathrm{d}\exp_X &= \mathrm{d}L_{\exp X}(e) \circ \frac{1-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)} \\
\Leftrightarrow \left.\frac{d}{dt}\right|_{t=0} \exp\left\{X + t\dot{X}\right\} &= e^X \frac{1-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}(X) := e^X \mathcal{T}_X(X)
\end{aligned}$$

This formula extends easily to super-groups – here Osp(2n|2n); the set where $\mathcal{T}_X^{-1}$ is well-defined is dense in the super-group. We continue setting $g = \exp X$, $h(t) = \exp tY \in$ Osp(2n|2n):

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} T^0_{gh(t)} &= \left.\frac{d}{dt}\right|_{t=0} T^0_{\exp(X)\exp(tY)} \\
&= \left.\frac{d}{dt}\right|_{t=0} T^0_{\exp\left(X+\mathcal{T}_X^{-1}(tY)\right)} \\
&= \left.\frac{d}{dt}\right|_{t=0} \exp\Big[\bar{c}^i_A \left(X + \mathcal{T}_X^{-1}(tY)\right)^{(11)}_{AB} c^i_B \\
&\quad + \bar{c}^i_A \left(X + \mathcal{T}_X^{-1}(tY)\right)^{(12)}_{AB} \bar{c}^i_B \\
&\quad + c^i_A \left(X + \mathcal{T}_X^{-1}(tY)\right)^{(21)}_{AB} c^i_B \\
&\quad + c^i_A \left(X + \mathcal{T}_X^{-1}(tY)\right)^{(22)}_{AB}\Big] \bar{c}^i_B \\
&= T_{\exp(X)} \left.\frac{d}{dt}\right|_{t=0} \exp\Big[\bar{c}^i_A \left(\mathcal{T}_X \circ \mathcal{T}_X^{-1}(tY)\right)^{(11)}_{AB} c^i_B \\
&\quad + \bar{c}^i_A \left(\mathcal{T}_X \circ \mathcal{T}_X^{-1}(tY)\right)^{(12)}_{AB} \bar{c}^i_B \\
&\quad + c^i_A \left(\mathcal{T}_X \circ \mathcal{T}_X^{-1}(tY)\right)^{(21)}_{AB} c^i_B \\
&\quad + c^i_A \left(\mathcal{T}_X \circ \mathcal{T}_X^{-1}(tY)\right)^{(22)}_{AB}\Big] \bar{c}^i_B \\
&= \left.\frac{d}{dt}\right|_{t=0} T^0_g T^0_{h(t)}
\end{aligned}$$

The homomorphism T^0_g on the vacuum subsector is thus – after integration – established. Since the form of the creator/annihilator pairs does not enter this proof it applies without substantial change to T^B_g [9].□

[9] To make this plausible, note that Lemma A.1 applies to any combination of creators and annihilators.

2.3.2 Construction of the Projector $\mathcal{P}$

The Isotropy Subgroup K According to [Zirn 96b] $K = \mathrm{Gl}(\mathrm{n}|\mathrm{n})$ is the isotropy subgroup of the ortho-symplectic group we will integrate over. A possible representation for this subgroup is the group of matrices $k = \mathrm{diag}(A,\, A^{-1T})$. The action of these elements stabilizes the vacuum, as expected:

$$\begin{aligned} T^0_k|0\rangle &= \exp\left(\bar{c}^i_A(\ln A)_{AB}c^i_B + c^i_A(\ln A^{-1T})_{AB}\bar{c}^i_B\right)|0\rangle \\ &= |0\rangle \exp\left(-N\sum_A(\ln A^{-1T})_{AA}\right) \\ &= |0\rangle \exp\left(\ \mathrm{STr}\ \ln(A^T)^{-N}\right) \\ &= |0\rangle\ \mathrm{SDet}\ A^{-N} \\ &\stackrel{\mathrm{def}}{=} |0\rangle\ \mu(k) \end{aligned}$$

Here $\mu(h)$ is a one-dimensional representation of the $\mathrm{Gl}(\mathrm{n}|\mathrm{n})$. In the same way we get:

$$\langle 0|\left(T^0_k\right)^{-1} = \mu^{-1}(k)\,\langle 0| \qquad (k \in K)$$

The same has to be true for the baryonic subsector as well:

$$T^B_{\mathrm{diag}(A,A^{-1T})}\bar{F}^B_{(e)}|0\rangle = \bar{F}^B_{(e)}|0\rangle \mathrm{SDet} A^{-N}$$

and also:

$$\langle 0|F^B_{(e)}T_{\mathrm{diag}(A,A^{-1T})} = \mathrm{SDet}A^N\langle 0|F^B_{(e)}.$$

The Parameterization of the Coset Space Now we need a parameterization of the coset space G/K. This can be done in terms of two supermatrices $Z, \tilde{Z}$. We start with a characterization of the $gH = Q \in G/K$ in terms of the $g \in G$. Taking $\Sigma_z = \sigma_z \otimes \mathbb{1}_{2n}$ we have:

$$\pi : g \mapsto Q = g\Sigma_z g^{-1} \tag{2.14}$$

a canonical projector from G to G/K, since k commutes with Σ_z:

$$k\Sigma_z = \Sigma_z k \ \Leftrightarrow\ \Sigma = k\Sigma_z k^{-1}$$

Assuming now the elements of G to be written in the creator/annihilator block decomposition already introduced

$$G \ni g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we take $Z = BD^{-1}$ and $\tilde{Z} = CA^{-1}$ and arrive at the Gaussian decomposition of g:

$$
\begin{aligned}
Q &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \\
&= \begin{pmatrix} (A + BD^{-1}C)(A - BD^{-1}C)^{-1} & -2B(D - CA^{-1}B)^{-1} \\ 2C(A - BD^{-1}C)^{-1} & -(D + CA^{-1}B)(D - CA^{-1}B)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{1} & Z \\ \tilde{Z} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & Z \\ \tilde{Z} & \mathbb{1} \end{pmatrix}^{-1}
\end{aligned}
$$

We decompose now $g = s(\pi(g))h(g)$ with π the projector above and $s : G/K \to G$ a section of the bundle G over G/K. This section can then be represented as:

$$
\begin{aligned}
s(Z, \tilde{Z}) &= \begin{pmatrix} (1 - Z\tilde{Z})^{-1} & Z(1 - \tilde{Z}Z)^{-1} \\ \tilde{Z}(1 - Z\tilde{Z})^{-1} & (1 - \tilde{Z}Z)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{1} & Z \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} (1 - Z\tilde{Z})^{\frac{1}{2}} & 0 \\ 0 & (1 - \tilde{Z}Z)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ \tilde{Z} & \mathbb{1} \end{pmatrix}
\end{aligned}
$$

This decomposition is said to "fix a gauge" in the following sense: When thinking of G as a fiber bundle with base G/K and fiber K, then π is the canonical projection from the fiber space onto the base. Now G can be seen as the field space of a gauge theory with a gauge group K. "Fixing the gauge" means then that we separate the unphysical gauge degrees of freedom from the physical ones over which we want to integrate. This can be achieved by choosing a smooth map $s : G/K \to G$ in such a way that $\pi \circ s = \mathrm{id}$. s distinguishes then some submanifold $\mathcal{M} \subset G$ which is then (locally) isomorphic to G/K.

We obtain the action – which is in fact an operator on $\mathcal{F}$ – of the Osp(2n|2n) on the Fock space by exponentiating the product of the block matrices with some bilinear combination of the super-creators and annihilators:

$$
\begin{aligned}
T^0_{s(\pi(g))} &= T^0_{\left(\begin{smallmatrix} 1 & Z \\ 0 & 1 \end{smallmatrix}\right)} T^0_{\left(\begin{smallmatrix} \ln(1+Z\tilde{Z})^{+1/2} & 0 \\ 0 & \ln(1+\tilde{Z}Z)^{-1/2} \end{smallmatrix}\right)} T^0_{\left(\begin{smallmatrix} 1 & 0 \\ \tilde{Z} & 1 \end{smallmatrix}\right)} \\
&= \exp\left(\bar{c}^i_A Z_{AB} \bar{c}^i_B\right) \exp\left(\bar{c}^i_A \frac{1}{2} \ln(1 - Z\tilde{Z})_{AB} c^i_B - \right. \\
&\qquad \left. - c^i_A \frac{1}{2} \ln(1 - \tilde{Z}Z)_{AB} \bar{c}^i_B\right) \exp\left(c^i_A \tilde{Z}_{AB} c^i_B\right)
\end{aligned}
$$

For the baryonic action we get in the same way:

$$
T^B_{s(\pi(g))} = \exp\left\{ \bar{c}^i_{\hat{A}} Z_{\hat{A}\hat{B}} c^i_{\hat{B}} + \bar{c}^i_{\hat{A}} Z_{\hat{A}(e,F)} \bar{f}^i_e + f^i_e Z_{(e,F)\hat{B}} c^i_{\hat{B}} + f^i_e Z_{(e,F)(e,F)} \bar{f}^i_e \right\} \times
$$

$$\times \exp \frac{1}{2} \left\{ \bar{c}^i_{\hat{A}} \left(\ln(1 - Z\tilde{Z}) \right)_{\hat{A}\hat{B}} c^i_{\hat{b}} + \bar{c}^i_{\hat{A}} \left(\ln(1 - Z\tilde{Z}) \right)_{\hat{A}(e,F)} \bar{f}^i_e + \right.$$
$$\left. + f^i_{(e,F)} \left(\ln(1 - Z\tilde{Z}) \right)_{(e,F)\hat{B}} c^i_{\hat{A}} + f^i_e \left(\ln(1 - Z\tilde{Z}) \right)_{(e,F)(e,F)} \bar{f}^i_e \right\} \times$$
$$\times \exp \left\{ c^i_{\hat{A}} \tilde{Z}_{\hat{A}\hat{B}} \bar{c}^i_{\hat{B}} + c^i_{\hat{A}} \tilde{Z}_{\hat{A}(e,F)} f^i_e + \bar{f}^i_e \tilde{Z}_{(e,F)\hat{B}} \bar{c}^i_{\hat{B}} + \bar{f}^i_e \tilde{Z}_{(e,F)(e,F)} f^i_e \right\}$$

When acting with T^0_g on the vacuum $|0\rangle$ and T^B_g on the Fermi-baryonic state $\bar{F}^B_{(e)}|0\rangle$ we arrive at a set of generalized coherent states[10] $|Z\rangle_0$ and $|Z\rangle_B$, as can be seen by expansion in power series.

$$\begin{aligned}
T^0_{s(Z,\tilde{Z})}|0\rangle &= \exp\left(\bar{c}^i_A Z_{AB} \bar{c}^i_B\right)|0\rangle \ \mathrm{SDet}\left(1 - \tilde{Z}Z\right)^{N/2} \stackrel{\mathrm{def}}{=} |Z\rangle_0 \\
\langle 0|T^{0\,-1}_{s(Z,\tilde{Z})} &= \mathrm{SDet}\left(1 - \tilde{Z}Z\right)^{N/2} \langle 0| \exp\left(-c^i_A \tilde{Z}_{AB} c^i_B\right) \stackrel{\mathrm{def}}{=} {}_0\langle Z|
\end{aligned}$$

$$\begin{aligned}
T^B_{s(Z,\tilde{Z})} \bar{F}^B_{(e)}|0\rangle &= \exp\left(\bar{c}^i_{\hat{A}} Z_{\hat{A}\hat{B}} \bar{c}^i_{\hat{B}} + \bar{c}^i_{\hat{A}} Z_{\hat{A}(e,F)} f^i_e + f^i_e Z_{(e,F)\hat{A}} \bar{c}^i_{\hat{A}} + \right. \\
&\quad \left. + f^i_e Z_{(e,F)(e,F)} f^i_e\right) \times \bar{F}^B_{(e)}|0\rangle \mathrm{SDet}\left(1 - \tilde{Z}Z\right)^{N/2} \stackrel{\mathrm{def}}{=} |Z\rangle_B \\
\langle 0|F^B_{(e)} T^{TD\,-1}_{s(Z,\tilde{Z})} &= \mathrm{SDet}\left(1 - \tilde{Z}Z\right) \langle 0|F^B_{(e)} \exp\left(-c^i_{\hat{A}} \tilde{Z}_{\hat{A}\hat{B}} c^i_{\hat{B}} - c^i_{\hat{A}} \tilde{Z}_{\hat{A}(e,F)} \bar{f}^i_e - \right. \\
&\quad \left. - \bar{f}^i_e \tilde{Z}_{(e,F)\hat{B}} c^i_{\hat{B}} - f^i_e \tilde{Z}_{(e,F)(e,F)} f^i_e\right) \stackrel{\mathrm{def}}{=} {}_B\langle Z|
\end{aligned}$$

We are now ready to define the projector $\mathcal{P}$ on the flavor sector:

$$\mathcal{P} \stackrel{\mathrm{def}}{=} \int_{\mathcal{M}_B \times \mathcal{M}_F} Dg_H T^0_g |0\rangle\langle 0| T^{0\,-1}_g + \int_{\mathcal{M}_B \times \mathcal{M}_F} Dg_H T^B_g \bar{F}^B_{(e)}|0\rangle\langle 0| F^B_{(e)} T^{B\,-1}_g \qquad (2.15)$$

Theorem 2.4 *The projector $\mathcal{P}$ in (2.15) is identical to unity on the flavor sector and vanishes elsewhere. Hence $\mathcal{P}$ projects the super-Fock space onto the flavor sector.*

Proof: Due to the translation invariance of the integration measure under $gH \mapsto g_0 gH$ (cf. [Zirn 96a]) we have:

$$\begin{aligned}
T_{g_0} \mathcal{P}^{\text{sector}} &= \int Dg_H T_{g_0} T_g |\text{state}\rangle\langle\text{state}| T^{-1}_g \\
&= \int Dg_H T_{g_0 g} |\text{state}\rangle\langle\text{state}| T^{-1}_g \\
&= \int Dg_H T_g |\text{state}\rangle\langle\text{state}| T^{-1}_{g_0^{-1} g} \\
&= \int Dg_H T_g |\text{state}\rangle\langle\text{state}| T^{-1}_g T_{g_0} = \mathcal{P}^{\text{sector}} T_{g_0}
\end{aligned}$$

where $|\text{state}\rangle$ is in $\left\{|0\rangle, \bar{F}^B|0\rangle\right\}$. Since the action of the Osp on each of the subsectors of the flavor sector is irreducible and every T_{g_0} commutes with its

[10] cf. [Pere 86]

corresponding $\mathcal{P}^{\text{sector}}$, we have then by Schur's lemma that each $\mathcal{P}^{\text{sector}}$ is a projector on its subsector proportional to the identity on this very subsector and zero elsewhere. Thus the sum $\mathcal{P} = \mathcal{P}^0 + \mathcal{P}^B$ is a projector on the flavor sector. To calculate now the constant of proportionality we continue as follows:

By taking $T_g = T_{s(\pi(g))} T_{k(g)}$ we get

$$\begin{aligned} T_g|\text{state}\rangle &= T_{s(\pi(g))} T_{k(g)}|\text{state}\rangle = T_{s(\pi(g))}|\text{state}\rangle \mu\left(k(g)\right) \\ \langle\text{state}|T_g^{-1} &= \mu^{-1}\left(k(g)\right) \langle\text{state}|T^{-1}_{s(\pi(g))} \end{aligned}$$

and thus

$$T_g|\text{state}\rangle\langle\text{state}|T_g^{-1} = T_{s(\pi(g))}|\text{state}\rangle\langle\text{state}|T^{-1}_{s(\pi(g))}. \tag{2.16}$$

which lets us arrive at

$$\begin{aligned} \mathcal{P} &= \int Dg_H T_g^0|0\rangle\langle 0|T_g^{0-1} + \int Dg_H T_g^B \bar{F}^B|0\rangle\langle 0|F^B T_g^{B-1} \\ &= \int D(Z,\tilde{Z})|Z\rangle_{0\,0}\langle Z| + \int D(Z,\tilde{Z})|Z\rangle_{B\,B}\langle Z|. \end{aligned} \tag{2.17}$$

By calculation of the vacuum and baryonic expectation value we can extract the constant of proportionality.

$$\begin{aligned} \langle 0|\mathcal{P}|0\rangle &= \int D(Z,\tilde{Z})\langle 0|Z\rangle_{0\,0}\langle Z|0\rangle = \int D(Z,\tilde{Z})\ \text{SDet}(1-\tilde{Z}Z)^N \\ &= \int D\mu_N(Z,\tilde{Z}) = 1 \end{aligned}$$

$$\begin{aligned} \langle 0|F^B\mathcal{P}\bar{F}^B|0\rangle &= \int D(Z,\tilde{Z})\langle 0|F^B|Z\rangle_{B\,B}\langle Z|\bar{F}^B|0\rangle \\ &= \int Dg_H\ \text{SDet}(1-\tilde{Z}Z)^N = \int D\mu_N(Z,\tilde{Z}) = 1 \end{aligned}$$

Obviously the expectation value of $\mathcal{P}^0$ vanishes on the baryonic subsector whereas $\mathcal{P}^B$ yields zero expectation value on the vacuum subsector. Therefore we can conclude that $\mathcal{P}$ is actually the identity on the flavor sector. Since the generalized coherent states are color singlet ${}_{\text{subsector}}\langle Z|$ vanishes outside its subsector – cf. lemma 2.2. □

2.3.3 The Bose-Fermi Coherent States

The Bose-Fermi coherent states are defined as being generated by $\exp\left(\bar{c}^i_A \psi^i_A\right)$. Here the ψ's are super-variables, that are supposed to fulfill the usual super-commutation relations:

$$\bar{c}^i_A \psi^i_A = (-1)^{|A|} \psi^i_A \bar{c}^i_A$$

The states generated thus from the vacuum obviously cover the entire Fock space. In fact they form the Grassmann envelope (cf. [Bere 87]) of the $\mathbb{Z}_2$ graded space of bosonic and fermionic many-particle states.

These states can be easily projected on our flavor sector by means of orthogonal rotations $\bar{c}^i_A \mapsto O^{ij}\bar{c}^i_A$ in color space. Averaging over all such rotations we arrive at a representation of the projector $\mathcal{P}$ on the flavor space[11]:

$$\mathcal{P} \exp\left(\bar{c}^i_A \psi^i_A\right)|0\rangle = \int_{\mathrm{SO(N)}} dO \exp\left(O^{ij}\bar{c}^j_A \psi^i_A\right)|0\rangle \tag{2.18}$$

Here dO is the (translation invariant) Haar-measure on the special orthogonal group (cf. [Rich 81]). To see that (2.18) is true consider the following:

Take $T'_O = \exp\left\{(\ln O)^{ij}\hat{F}^{ij}\right\}$ and write then[12]:

$$\mathcal{P} \exp\left(\bar{c}^i_A \psi^i_A\right)|0\rangle = \int_{\mathrm{SO(N)}} dO\, T'_O \exp \bar{c}^i_A \psi^i_A |0\rangle \tag{2.19}$$

We have then if $|\mathrm{state}\rangle \in \{|\mathrm{flavor}\rangle\}$:

$$\mathcal{P}|\mathrm{flavor}\rangle = |\mathrm{flavor}\rangle,$$

since this is true for each $T'_O|\mathrm{flavor}\rangle = |\mathrm{flavor}\rangle$. For an arbitrary state $|\mathrm{state}\rangle$ and $T'_{\tilde{O}} = \exp\{s \cdot \hat{F}^{ij}\}$ by the translation invariance of dO:

$$\mathcal{P}|\mathrm{state}\rangle = T'_{\tilde{O}}\mathcal{P}|\mathrm{state}\rangle$$

and after differentiation by s:

$$\hat{F}^{ij}\mathcal{P}|\mathrm{state}\rangle = 0$$

thereby proving that $\mathcal{P}$ is really a projector onto the flavor sector.

[11]In fact, this is actually a way to define this projector.

[12]This equality is proven in detail in appendix C.

2.4 The Proof

As we now have the two expressions for the projector at hand we can finally prove the color-flavor transformation (2.1):

$$
\begin{aligned}
& \int_{\mathrm{SO(N)}} dO \, \exp\left(\bar{\psi}^i_A O^{ij} \psi^j_A\right) \\
= \; & \int_{\mathrm{SO(N)}} dO \, \langle 0| \exp\left(\bar{\psi}^i_A c^i_A\right) \exp\left(\bar{c}^i_A O^{ij} \psi^j_A\right) |0\rangle \\
= \; & \langle 0| \exp\left(\bar{\psi}^i_A c^i_A\right) \mathcal{P} \exp\left(\bar{c}^i_A \psi^i_A\right) |0\rangle \\
= \; & \langle 0| \exp\left(\bar{\psi}^i_A c^i_A\right) \left(\int D(Z,\tilde{Z}) |Z\rangle_{0\,0}\langle Z| + \int D(Z,\tilde{Z}) |Z\rangle_{B\,B}\langle Z| \right) \times \\
& \quad \times \exp\left(\bar{c}^i_A \psi^i_A\right) |0\rangle \\
= \; & \int D\mu_N(Z,\tilde{Z}) \, \langle 0| \exp\left(\bar{\psi}^i_A c^i_A\right) \exp\left(\bar{c}^i_A Z_{AB} \bar{c}^i_B\right) |0\rangle\langle 0| \exp\left(-c^i_A \tilde{Z}_{AB} c^i_B\right) \times \\
& \quad \times \exp\left(\bar{c}^i_A \psi^i_A\right) |0\rangle + \\
& + \int D\mu_N(Z,\tilde{Z}) \, \langle 0| \exp\left(\bar{\psi}^i_A c^i_A\right) \exp\left(\bar{c}^i_{\hat{A}} Z_{\hat{A}\hat{B}} \bar{c}^i_{\hat{B}} + \bar{c}^i_{\hat{A}} Z_{\hat{A}(e,F)} f^i_e + f^i_e Z_{(e,F)\hat{B}} \bar{c}^i_{\hat{B}}\right) \times \\
& \quad \times \bar{F}^B_{(e)} |0\rangle\langle 0| F^B_{(e)} \exp\left(c^i_{\hat{A}} \tilde{Z}_{\hat{A}\hat{B}} c^i_{\hat{B}} + c^i_{\hat{A}} \tilde{Z}_{\hat{A}(e,F)} \bar{f}^i_e + \bar{f}^i_e \tilde{Z}_{(e,F)\hat{B}} c^i_{\hat{B}}\right) \exp\left(\bar{c}^i_A \psi^i_A\right) |0\rangle \\
= \; & \int D\mu_N(Z,\tilde{Z}) \exp\left(\bar{\psi}^i_A Z_{AB} \bar{\psi}^i_B + \psi^i_A \tilde{Z}_{AB} \psi i_B\right) + \\
& + \int D\mu_N(Z,\tilde{Z}) \exp\left(\bar{\psi}^i_{\hat{A}} Z_{\hat{A}\hat{B}} \bar{\psi}^i_{\hat{B}} + \bar{\psi}^i_{\hat{A}} Z_{\hat{A}(e,F)} \psi^i_{(e,F)} + \psi^i_{(e.F)} Z_{(e,F)\hat{B}} \bar{\psi}^i_{\hat{B}} + \right. \\
& \quad \left. + \psi^i_{\hat{A}} \tilde{Z}_{\hat{A}\hat{B}} \psi^i_{\hat{B}} + \psi^i_{\hat{A}} \tilde{Z}_{\hat{A}(e,F)} \bar{\psi}^i_{(e,F)} + \bar{\psi}^i_{(e,F)} \tilde{Z}_{(e,F)\hat{B}} \psi^i_{\hat{B}}\right) \qquad \square
\end{aligned}
\tag{2.20}
$$

Chapter 3

Random Bond Ising Model

In this chapter we will derive a non-linear sigma model for the random-bond Ising model. This is achieved by mapping the RBI onto a network model which is similar to the Chalker-Coddington model. Taking the continuum limit of the network model we would arrive at the non-linear sigma model.

3.1 The Network Model

In this section we give a brief review of how to arrive at the network model. This procedure was outlined in an article by Cho and Fisher [Cho 97]. More information can be gained from this paper and from [Ho 96] and [Lee 94].

We start with an interaction Hamiltonian similar to the classical 2D Ising model:

$$H = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i^3\sigma_j^3 \tag{3.1}$$

The coupling constant for the link (i,j), J_{ij}, can have a positive or a negative sign while the absolute value of J is the same for all links. By taking an anisotropic continuum limit in one direction (which is referred to as the "time" direction; cf. [Kogu 79]) we arrive at a 1D time-continuum equivalent representation:

$$H = \sum \iota_1\sigma_n^1 + \iota_2\sigma_n^3\sigma_{n+1}^3$$

A change in the signs of all J_{ij} in the Hamiltonian (3.1) corresponds to a change of sign for $\iota_{1,2}$. Thus setting $\iota = \iota(n,\tau)$ – making it a function of space and time – incorporates the randomness.

To arrive at a fermionic representation we introduce the Majorana fields

$$\eta_1(n) = \frac{1}{\sqrt{2}}\prod_{m<n}\sigma_m^1\sigma_n^2, \qquad \eta_2(n) = \frac{1}{\sqrt{2}}\prod_{m<n}\sigma_m^1\sigma_n^3$$

that anticommute:

$$\begin{aligned}
\{\eta_1(n),\eta_2(n')\} &= \frac{1}{2}\left(\prod_{m<n}\sigma^1_m\prod_{m'<n'}\sigma^1_{m'}\sigma^2_n\sigma^3_{n'}+\prod_{m'<n'}\sigma^1_{m'}\prod_{m<n}\sigma^1_m\sigma^3_{n'}\sigma^2_n\right)\\
&\overset{\substack{\text{assume}\\ n<n'}}{=} \frac{1}{2}\left(\prod_{n\le m'<n'}\sigma^1_{m'}\sigma^2_n\sigma^3_{n'}+\prod_{n\le m'<n'}\sigma^1_{m'}\sigma^3_{n'}\sigma^2_n\right)\\
&= \frac{1}{2}\prod_{n\le m'<n'}\sigma^1_{m'}\{\sigma^2_n,\sigma^3_{n'}\}=0
\end{aligned}$$

We can now express the Hamiltonian in terms of these fields:

$$H=(-2i)\sum_n[\iota_1\eta_1(n)\eta_2(n)-\iota_2\eta_1(n)\eta_2(n+1)]$$

since

$$\begin{aligned}
\eta_1(n)\eta_2(n) &= \frac{1}{2}\prod_{m<n}\sigma^1_m\sigma^2_n\prod_{m'<n}\sigma^1_{m'}\sigma^3_n\\
&= \frac{1}{2}\sigma^2_n\sigma^3_n=\frac{i}{2}\sigma^1_n
\end{aligned}$$

and in the same way

$$\begin{aligned}
\eta_1(n)\eta_2(n+1) &= \frac{1}{2}\sigma^2_n\sigma^1_n\sigma^3_{n+1}\\
&= \frac{-i}{2}\sigma^3_n\sigma^3_{n+1}.
\end{aligned}$$

Taking another set of (independent) Majorana fields $\xi_1(n),\xi_2(n)$ and taking $\psi_i=\frac{1}{\sqrt{2}}(\eta_i+i\xi_i)$ we arrive at a Dirac fermion representation with

$$\begin{aligned}
H_{\text{Dirac}} &= \sum_n(-i\iota_1)\left[\psi_1^\dagger(n)\psi_2(n)-\psi_2^\dagger(n)\psi_1(n)\right]+\\
&\quad +(i\iota_2)\left[\psi_1^\dagger(n)\psi_2(n+1)-\psi_2^\dagger(n+1)\psi_1(n)\right].
\end{aligned}$$

The mixed terms in η and ξ cancel precisely, as expected.

Writing then the partition function in terms of a functional integral over Grassmann fields we get

$$Z=\int D(\psi,\bar\psi)\exp(-S)$$

with the action

$$S = \int_\tau \sum_n [\bar{\psi}_1(n)\partial_\tau \psi_1(n) + \bar{\psi}_2(n)\partial_\tau \psi_2(n)] + H_{\text{Dirac}}(\psi, \bar{\psi}).$$

When reinterpreting time as a (continous) spatial coordinate and trading S for a 2D Hamiltonian of (chiral) fermions, we arrive at a picture of a stack of 1D right/left movers:

$$\begin{aligned} H_{2D} \;=\; & \int dx \sum_n \psi^\dagger_{Rn}(i\partial_x)\psi_{Rn} + \psi^\dagger_{Ln}(i\partial_x)\psi_{Ln} + \\ & + \iota_1(\psi^\dagger_{Rn}\psi_{Ln} + \psi^\dagger_{Ln}\psi_{Rn}) + \\ & + \iota_2(\psi^\dagger_{Rn}\psi_{Ln+1} + \psi^\dagger_{Ln+1}\psi_{Rn}) \end{aligned} \tag{3.2}$$

with the chiral fermions

$$\begin{aligned} \psi_{Rn} &= (-1)^n \psi_1(n) \qquad & \psi^\dagger_{Rn} &= i(-1)^n \bar{\psi}_1(n) \\ \psi_{Ln} &= (-1)^n \psi_2(n) \qquad & \psi^\dagger_{Ln} &= i(-1)^n \bar{\psi}_2(n). \end{aligned}$$

This Hamiltonian corresponds to the situation depicted in figure 3.1. The energy eigenvalues are given by

$$E^2 = p_x^2 + \iota_1^2 + \iota_2^2 + 2\iota_1\iota_2 \cos p$$

with p_x the x-component of the momentum and p the transverse momentum – $p \in [-\pi; \pi]$. The energy is minimal for $p_x = 0$ and $p = \pi$. It is then denoted by $E_{\text{min}} = \pm|\Delta|$ with $\Delta = \iota_1 - \iota_2$. An incident wave with $E = 0$ will decay with $\exp(-|\Delta|x)$. The decay length $\xi \sim |\Delta|^{-1}$ approaches infinity for the pure Ising model with $\Delta = 0$. This clearly describes the extended states and the long range order at the critical point. The critical exponent is then – as would be expected – $\nu = 1$.

Figure 3.1 can now be interpreted in terms of a Chalker-Coddington network model. This is done pictorialy in figure 3.2 (a). To establish a connection between our ι s and the θ s of the network model we take a closer look at figure 3.2 (b): Following [Chal 88] we assign a transfer matrix to the node $P(00)$.

$$\begin{pmatrix} \phi_{P(00);1} \\ \phi_{P-e_x(01);2} \end{pmatrix} = \begin{pmatrix} \cosh\theta_1 & \sinh\theta_1 \\ \sinh\theta_1 & \cosh\theta_1 \end{pmatrix} \begin{pmatrix} \phi_{P(01);1} \\ \phi_{P(00);2} \end{pmatrix}$$

This transfer matrix conserves the current; it can be rewritten as a S-matrix and takes the form

$$\begin{pmatrix} \phi_{P(00);1} \\ \phi_{P(00);2} \end{pmatrix} = \frac{1}{\cosh\theta_1} \begin{pmatrix} 1 & -\sinh\theta_1 \\ \sinh\theta_1 & 1 \end{pmatrix} \begin{pmatrix} \phi_{P(01);1} \\ \phi_{P-e_x(01);2} \end{pmatrix}. \tag{3.3}$$

Since the tunneling probability in (3.2) is proportional to ι_i^2 we can identify $\tanh(\theta_i)$ with ι_i.

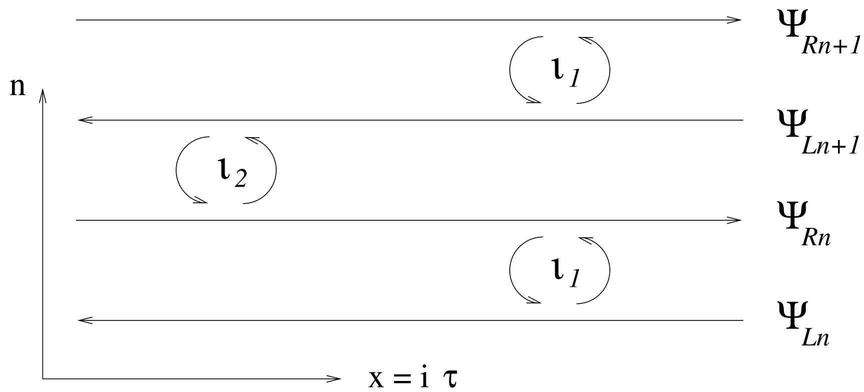

Figure 3.1: Pictorial representation of eqn. (3.2)

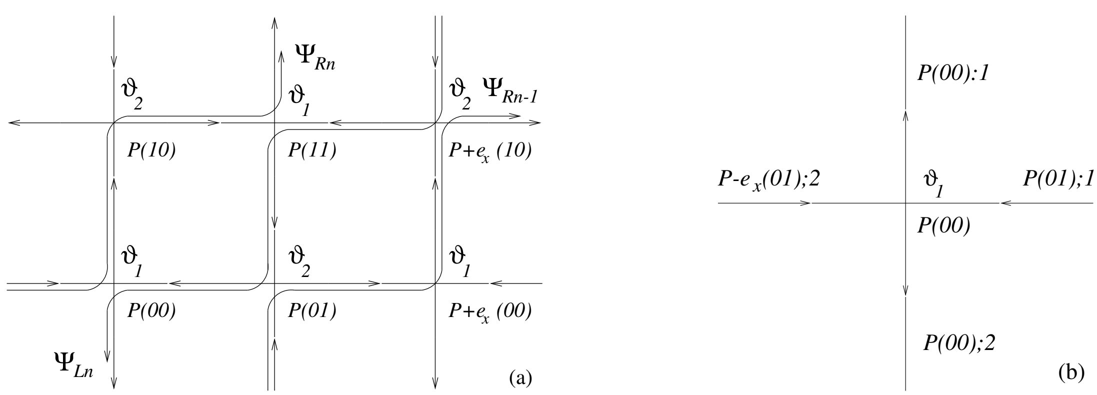

Figure 3.2: (a) Mapping of (3.2) onto a CC network model. (b) A close-up of a single node.

3.2 Introducing Disorder

As already explained in sec. 3.1 and in the introduction, disorder enters with the sign of the coupling constant J. This O(1) randomness is transfered to the network model, where it corresponds to a randomness in the sign of the θ s. Since $\cosh\theta$ is an even function in the argument θ we have no randomness in those terms, but only in those with $\tanh\theta$.

To arrive at a non-linear sigma model we proceed by calculating the disorder averaged expectation value for an operator that bears some resemblance to the conductance operator in the Chalker Coddington network model. This is justified by the fact that we expect only one relevant length scale for two-point correlation functions. The spin-correlation length for the 2D Hamiltonian (3.2) should therefore behave in the same way as the decay length ξ for the conductance g. M and N denote in the following two specific links between which the conductance is to be calculated. To arrive at an expression for g we follow the approach of [Zirn 97] for the original Chalker-Coddington model.

When taking two arbitrary links within the network and trying to calculate the averaged mean conductance by the Landauer-Büttiker formula $g_{MN} = |S_{MN}|^2$, we need to calculate an expression for the S-matrix S_{MN}. We are then interested in the stationary states. They are determined by the Schrödinger equation $U\psi = \lambda\psi$ with $U = \exp(iHt)$ the time evolution operator. While the eigenphases λ can be gauged away, we still have to solve the remaining equation. This is done by iteration; we obtain

$$g_{MN} = |S_{MN}|^2 = \left|\langle l_M|\tilde{T}|l_N\rangle\right|^2$$

with $\tilde{T} = U + UU'U + UU'UU'U + \ldots = (1 - UU')^{-1}U'$ where U denotes the one-step operator living in link-space that governs how the probability amplitudes are scattered at the nodes and U' denotes the phase that is picked up during the transmission of a link[1]. The operator $\tilde{T}$ describes how the complete lattice evolves over time. For our convenience we drop the last U in the following calculations. This will not change the behavior dramatically since we loose only the last scattering process. Thus we finally arrive at the operator $T = (1 - U)^{-1}$ that we will use to construct the sigma-model.

3.2.1 Rewriting the Matrix Elements

To proceed we need to express the matrix element $\langle l_M|T|l_N\rangle$ and its conjugate in terms of a Gaussian integral over super-numbers ϕ (cf. section 1.2.3). Thus we get:

[1] U' is obviously diagonal in link-space and has the same value for all links. We thus can take it to be the identity operator and drop it in the subsequent discussion.

$$
\begin{aligned}
&\langle l_N|(1-U^\dagger)^{-1}|l_M\rangle\langle l_M|(1-U)|l_N\rangle \\
&\quad = \int\prod_{p,t} D\left(\phi,\bar\phi\right)\phi_{-B}(l_N)\bar\phi_{-B}(l_M)\phi_{+B}(l_M)\bar\phi_{+B}(l_N)\times \\
&\qquad \times\exp\left\{-\sum_{p,t,p',t',\sigma}\bar\phi_{+\sigma}(p';t')\left[\delta\left((p';t'),(p;t)\right)-U\left((p';t'),(p;t)\right)\right]\phi_{+\sigma}(p;t)\right\} \\
&\qquad \times\exp\left\{-\sum_{p,t,p',t',\sigma}\bar\phi_{-\sigma}(p';t')\left[\delta\left((p';t'),(p;t)\right)-U^\dagger\left((p';t'),(p;t)\right)\right]\phi_{-\sigma}(p;t)\right\}
\end{aligned}
\tag{3.4}
$$

For clarity we have dropped the Einstein summation convention. The integration measure is written in shorthand and is supposed to stand for the "flat" Berezin measure over all ϕ s on all links in all flavors:

$$
\begin{aligned}
&\prod_{p,t} D\left(\phi,\bar\phi\right) \\
&\quad = \prod_{p,t} D\left(\phi_{\pm\sigma}(p;t),\bar\phi_{\pm\sigma}(p;t)\right) \\
&\quad = \prod_{P,t} D\left(\phi_{\pm\sigma}(P(00);t),\ldots,\phi_{\pm\sigma}(P(11);t),\bar\phi_{\pm\sigma}(P(00);t),\ldots,\bar\phi_{\pm\sigma}(P(11);t)\right)
\end{aligned}
$$

Again: for notational convenience we have introduced an alternative system of labeling the nodes: When referring to a sum over all nodes we replace the $P(00),\ldots$ notation by p, which is supposed to run over all nodes.

3.2.2 Performing the Disorder Average for $p=\frac{1}{2}$

The next step is to take the disorder in the coupling into account. We therefore have to average over the O(1) group when taking $p=\frac{1}{2}$ in equation (1.2). To do so, we obviously need the precise dependence of the operator U on the group elements. Since U can only couple to two adjacent links, we get the following expression:

$$
\begin{aligned}
U^{(1)}_{p''}\left((p';t'),(p;t)\right) &= \frac{J}{\cosh\theta_1}\delta(p',p'')\Big\{\Big[\delta(t';1)\delta(t;2)\delta(p''+e_y,p)+ \\
&\quad +\delta(t',2)\delta(t,1)\delta(p''-e_y,p)\Big] \\
&\quad +O_p\sinh\theta_1\Big[\delta(t',1)\delta(t,1)\delta(p''-e_y,p)- \\
&\quad -\delta(t',2)\delta(t,2)\delta(p''+e_y,p)\Big]\Big\}
\end{aligned}
$$

Here p'' takes only values in the nodes that are actually occupied by a θ_1. This can be read off from eqn. (3.3) and figure 3.2 (b). A very similar expression holds for θ_2 sites. The (node-dependant) O_p is the O(1) factor, i.e. takes values in $\{+1;-1\}$.

We are now in the position of being able to apply the color-flavor transformation (2.1) to this situation, thus taking the disorder average from an integration over the O(1) to an integration over the homogenous super-space. The fact that we integrate over the O(1) – and not the SO(1) – leads us to lose the baryonic term in (2.1). Thus we get after the transformation:

$$\begin{aligned}
&\int \prod_p D\mu\left(Z(p), \tilde{Z}(p)\right) \int \prod_{p,t} D\left(\phi(p,t), \bar{\phi}(p,t)\right) \phi_{-B}(l_N)\bar{\phi}_{-B}(l_M)\phi_{+B}(l_M)\bar{\phi}_{+B}(l_N)\times \\
&\times \exp\Bigg\{-\sum_{P,t,\tau=\pm,\sigma} \bar{\phi}_{\tau\sigma}\left(P(00);t\right)\phi_{\tau\sigma}\left(P(00);t\right) + \bar{\phi}_{\tau\sigma}\left(P(11);t\right)\phi_{\tau\sigma}\left(P(11);t\right) + \\
&\quad +\bar{\phi}_{\tau\sigma}\left(P(01);t\right)\phi_{\tau\sigma}\left(P(01);t\right) + \bar{\phi}_{\tau\sigma}\left(P(10);t\right)\phi_{\tau\sigma}\left(P(10);t\right) + \\
&+J\sum_{P,\tau=\pm,\sigma} \frac{1}{\cosh\theta_1}\Big[\bar{\phi}_{\tau\sigma}\left(P(00);1\right)\phi_{\tau\sigma}\left(P-e_x(10);2\right) + \bar{\phi}_{\tau\sigma}\left(P(00);2\right)\phi_{\tau\sigma}\left(P(10);1\right) + \\
&\quad + \bar{\phi}_{\tau\sigma}\left(P(11);1\right)\phi_{\tau\sigma}\left(P(01);2\right) + \bar{\phi}_{\tau\sigma}\left(P(11);2\right)\phi_{\tau\sigma}\left(P+e_x(01);1\right)\Big] + \\
&\quad +\frac{1}{\cosh\theta_2}\Big[\bar{\phi}_{\tau\sigma}\left(P(01);1\right)\phi_{\tau\sigma}\left(P+e_y(00);2\right) + \bar{\phi}_{\tau\sigma}\left(P(01);2\right)\phi_{\tau\sigma}\left(P(00);1\right) + \\
&\quad + \bar{\phi}_{\tau\sigma}\left(P(10);1\right)\phi_{\tau\sigma}\left(P(11);2\right) + \bar{\phi}_{\tau\sigma}\left(P(10);2\right)\phi_{\tau\sigma}\left(P-e_y(11);1\right)\Big] + \\
&+J\sum_{\substack{P,t,t' \\ \tau=\pm,\sigma,\sigma'}} \tanh\theta_1\Big[\bar{\phi}_{\tau\sigma}\left(P(00);t\right) Z^{P(00)}_{\sigma t\sigma' t'}\bar{\phi}_{\tau\sigma'}\left(P(00);t'\right) + \bar{\phi}_{\tau\sigma}\left(P(11);t\right) Z^{P(11)}_{\sigma t\sigma' t'}\bar{\phi}_{\tau\sigma'}\left(P(11);t'\right) + \\
&\quad +\phi_{\tau\sigma}\left(P-(t-1)e_x(10);t\right)\tilde{Z}^{P(00)}_{\sigma t\sigma' t'}\phi_{\tau\sigma'}\left(P-(t'-1)e_x(10);t'\right) + \\
&\quad + \phi_{\tau\sigma}\left(P+(2-t)e_x(01);t\right)\tilde{Z}^{P(11)}_{\sigma t\sigma' t'}\phi_{\tau\sigma'}\left(P+(2-t')e_x(01);t'\right)\Big] + \\
&\quad + \tanh\theta_2\Big[\bar{\phi}_{\tau\sigma}\left(P(01);t\right) Z^{P(01)}_{\sigma t\sigma' t'}\bar{\phi}_{\tau\sigma'}\left(P(01);t'\right) + \bar{\phi}_{\tau\sigma}\left(P(10);t\right) Z^{P(10)}_{\sigma t\sigma' t'}\bar{\phi}_{\tau\sigma'}\left(P(10);t'\right) + \\
&\quad +\phi_{\tau\sigma}\left(P+(t-1)e_y(00);t\right)\tilde{Z}^{P(01)}_{\sigma t\sigma' t'}\phi_{\tau\sigma'}\left(P+(t'-1)e_y(00);t'\right) + \\
&\quad + \phi_{\tau\sigma}\left(P-(2-t)e_y(11);t\right)\tilde{Z}^{P(10)}_{\sigma t\sigma' t'}\phi_{\tau\sigma'}\left(P-(2-t')e_y(11);t'\right)\Big]\Bigg\}
\end{aligned} \tag{3.5}$$

We are now ready to integrate out the super-vectors. Since we have terms of form $\bar{\phi}\bar{\phi}$, $\bar{\phi}\phi$ and $\phi\phi$ we cannot use the ordinary Gauss formula, but have to apply the formula derived in appendix D. For this we have to write down the matrix M as introduced in the notation of the appendix. This is – in principle – no problem, but requires some careful notation for large lattices. We will see what M looks like in the case of a small lattice.

Example: A Single Plaquette

Consider a network consisting of a single plaquette as shown in figure 3.3. We can now write down the matrix M using equation (3.5).

$$M = \left(\begin{array}{cccc|cccc} \mathbb{1} & 0 & \frac{i\sigma_y}{\cosh\theta_2} & 0 & Z^{(00)} & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & \frac{i\sigma_y}{\cosh\theta_1} & 0 & Z^{(01)} & 0 & 0 \\ \frac{i\sigma_y}{\cosh\theta_1} & 0 & \mathbb{1} & 0 & 0 & 0 & Z^{(10)} & 0 \\ 0 & \frac{i\sigma_y}{\cosh\theta_2} & 0 & \mathbb{1} & 0 & 0 & 0 & Z^{(11)} \\ \hline \tilde{Z}^{(01)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{Z}^{(11)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{Z}^{(00)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{Z}^{(10)} & 0 & 0 & 0 & 0 \end{array}\right)$$

Here the σ_y s denote the standard Pauli matrices and the Z s include the tanh s and J s. The whole matrix is principially graded in $\psi, \bar{\psi}$ with the block matrices being denoted by A, B, C and D as usual. The smaller grading is in the node parameter $00, 01, \ldots$. The matrix entries are matrices in their own right and have entries for the interaction between the link types (i.e. 1, 2). For clarity we suppress the boson/fermion grading.

To obtain now the lattice action we have to apply corollary D.1. To be able to do so we must make M Osp symmetric. This is done by introducing a factor $\frac{1}{2}$ in block matrix A thus getting A' and then exchanging the ψ's and $\bar{\psi}$'s to arrive at a $D = -A'^T$ matrix. Furthermore we have to multiply the two lower block matrices with $-\sigma$. We have then

$$\langle g_{MN}\rangle = \int \prod_p D\mu\left(Z(p), \tilde{Z}(p')\right) \mathrm{SDet}^{-1} M' \, A'^{k_1(M)k_2(N)}_{BB} A'^{k_2(N)k_1(M)}_{BB}$$

We now have to note that

$$D\mu\left(Z(p), \tilde{Z}(p)\right) = \mathrm{SDet}\left(1 - Z(p)\tilde{Z}(p')\right).$$

From this we can read off the lattice field theory:

$$\langle \bullet \rangle = \int \prod_p D\left(Z(p), \tilde{Z}(p)\right) \bullet \exp\left\{-S_{\text{lattice}}[Z(p), \tilde{Z}(p)]\right\} \tag{3.6}$$

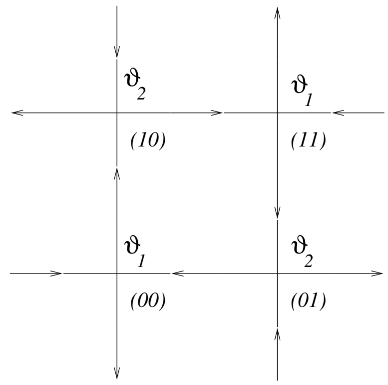

Figure 3.3: An easy example network to demonstrate the structure of M.

with lattice action

$$S_{\text{lattice}}[Z(p), \tilde{Z}(p)] = \ln \mathrm{SDet} M - \ln \mathrm{SDet}\left(1 - \tilde{Z}(p) Z(p)\right)$$

where the last term stems from the measure. In this discussion we have neglected terms that might occur through boundary conditions.

Obviously the lattice field theory derived in the example above can be generalized to arbitrary lattices, so that (3.6) still holds. The general form of M can be read off from (3.5). Writing it with Kronecker-deltas we get:

$$\begin{aligned}
A \; = \; & \delta_{XX'}\delta_{YY'}\delta_{xx'}\delta_{yy'}\delta_{tt'}\delta_{\sigma\sigma'} + \\
& + J(1-\delta_{tt'})\delta_{\sigma\sigma'}\left\{(\cosh\theta_1)^{-1}\delta_{YY'}\delta_{xy}\delta_{yy'}\left[\delta_{x'1}\delta_{x0}\left(\delta_{t1}\delta_{(X-1)X'} + \delta_{t2}\delta XX'\right) + \right.\right. \\
& \quad \left. + \delta_{X'0}\delta_{x1}\left(\delta_{t1}\delta_{XX'} + \delta_{t2}\delta_{(X+1)X'}\right)\right] + \\
& \quad + (\cosh\theta_2)^{-1}\delta_{XX'}\delta_{xx'}\delta_{x'y'}\left[\delta_{y1}\delta_{y'0}\left(\delta_{t1}\delta_{(Y+1)Y'} + \delta_{t2}\delta_{YY'}\right) + \right. \\
& \quad \left.\left. + \delta_{y0}\delta_{y'1}\left(\delta_{t1}\delta_{YY'} + \delta_{t2}\delta_{(Y-1)Y'}\right)\right]\right\} \\
Z \; = \; & J\,\delta_{XX'}\delta_{YY'}\delta_{xx'}\delta_{yy'}\delta_{XX''}\delta_{YY''}\delta_{xx''}\delta_{yy''}\,Z^{(X''Y'')(x''y'')}_{\sigma t\sigma' t'} \times \\
& \times\left[\tanh\theta_1\delta_{xy} + \tanh\theta_2(1-\delta_{xy})\right] \\
\tilde{Z} \; = \; & J\,\delta_{XX'}\delta_{YY'}\delta_{xx'}\delta_{yy'}\,Z^{(X''Y'')(x''y'')}_{\sigma t\sigma' t'} \times \\
& \times\left\{\tanh\theta_1\,(1-\delta_{xy})\delta_{x''y''}\delta_{yx''}\left[\delta_{x1}\left(\delta_{t1}\delta_{XX''} + \delta_{t2}\delta_{X(X''-1)}\right) + \right.\right.
\end{aligned}$$

$$+\delta_{x0}\left(\delta_{t1}\delta_{X(X''+1)}+\delta_{t2}\delta_{XX''}\right)\Big]+$$
$$+\tanh\theta_2\,\delta_{xy}(1-\delta_{x''y''})\delta_{xx''}\Big[\delta_{x1}\left(\delta_{t1}\delta_{X(X''-1)}+\delta_{t2}\delta_{XX''}\right)+$$
$$+\delta_{x0}\left(\delta_{t1}\delta_{XX''}+\delta_{t2}\delta_{X(X''+1)}\right)\Big]\Big\}$$

where X, Y are appropiate coordinates for P.

3.2.3 Arbitrary Probability Distribution

The results in the last section were obtained for $p=\frac{1}{2}$ in (1.2) only. To generalize these results we have to integrate over SO(1) in order to arrive at a sigma model for $p=1$. This is done in appendix E.

Having the sigma model for both – $p=1$ and $p=\frac{1}{2}$ – we can now construct a sigma model for arbitrary probabilities $p\in[0;1]$. Note therefore that

$$\begin{aligned}\langle\hat{A}(O)\rangle^p &= p\hat{A}(1)+(1-p)\hat{A}(-1)\\ &= 2(1-p)\left[\frac{\hat{A}(1)+\hat{A}(-1)}{2}\right]+(2p-1)\hat{A}(1).\end{aligned}\tag{3.7}$$

If we examine now $\langle\bullet\rangle^{p=1}$ and $\langle\bullet\rangle^{p=1/2}$ – corresponding to an integration over SO(1) viz. O(1) respectivly – we notice that

$$\langle\bullet\rangle^{p=1}=\langle\bullet\rangle^{p=1/2}+\langle\bullet\rangle^{B}$$

and further

$$\langle\bullet\rangle^{p\in[\frac{1}{2};1]}=\langle\bullet\rangle^{p=1/2}+\lambda\langle\bullet\rangle^{B}$$

with $\lambda\in[0;1]$. We note that

$$\langle\hat{A}(O)\rangle^{p=1/2}=\frac{\hat{A}(1)+\hat{A}(-1)}{2}= \quad\text{and}\quad \langle\hat{A}(O)\rangle^{p=1}=\hat{A}(1).$$

Comparing this to (3.7) we get for $p\in[\frac{1}{2};1]$:

$$\langle\bullet\rangle^{p}=\langle\bullet\rangle^{p=1/2}+\lambda\langle\bullet\rangle^{B}$$

with $\lambda=(2p-1)$.

Chapter 4

Conclusions and Outlook

In this work we obtained two important results – a mathematical and a physical one:

1. The color-flavor transformation for the SO(N),
2. The lattice action for the network model corresponding to the random bond Ising model with binary probability distribution.

The Color-Flavor Transformation for SO(N)

The mathematical result is expected to be applicable to a wide range of physical systems, since it is valid for all models of universality class of type D – i.e. those with SO(N) symmetry. Among other possible applications let us mention SNS-quantum dots at very low temperatures with time-reversal and spin-rotation invariance broken by magnetic fields viz. spin-orbit coupling.

The Lattice Field Theory for the RBI

The lattice action we derived is the starting point for a continuum limit for the network model. Basically this should be achieved in the same way as the continuum theory for the Chalker-Coddington network in [Zirn 97] for the integer QHE. Still, we expect some difficulties, since the corresponding Boltzmann weight $W[Z,\tilde{Z}] = \exp(-S_{\text{lattice}}) = \text{SDet} M^{-1} \text{SDet}(1 - \tilde{Z}Z)$ does not – at least trivially – factorize. This sigma-model – given in terms of the coordinates for the coset space $Z, \tilde{Z}$ – should in turn be evaluable, so that the critical exponents for the spin-correlation length along the phase boundary and along the Nishimori-line can be extracted. If that was possible, we could compare these results to those obtained numerically.

Appendix A

The osp(2n,2n) algebra

Consider the following set of generators:

$$\begin{array}{llll} B^{ab} = \bar{b}^i_a \bar{b}^i_b & F^{ab} = \bar{f}^i_a \bar{f}^i_b & G^{ab} = \bar{b}^i_a \bar{f}^i_b & \\ B^b_a = \frac{1}{2}\left(b^i_a \bar{b}^i_b + \bar{b}^i_b b^i_a\right) & F^b_a = \frac{1}{2}\left(f^i_a \bar{f}^i_b - \bar{f}^i_b f^i_a\right) & G^b_a = \bar{b}^i_a f^i_b & \tilde{G}^b_a = b^i_a \bar{f}^i_b \\ B_{ab} = b^i_a b^i_b & F_{ab} = f^i_a f^i_b & G_{ab} = b^i_a f^i_b & \end{array}$$

These operators are obviously color singlet and will therefore sometimes be denoted as $\hat{C}_{ab}$.

Theorem A.1 *The operators above form a representation of the* osp(2n,2n) *algebra.*

The linear structure property is rather obvious. To prove that they do indeed form an algebra, we have to show that some rule $(X, Y) \mapsto [X, Y]$ exists. It has to be bilinear and must fulfill the following properties: (i) $[X, X] = 0$, and (ii) the Jacobi identity. The choice of the super-commutator is a good one. We will show that the super-commutators remain within the algebra.

A.1 Preliminaries

In order to arrive at the representation for the generators given above we will need the following lemma:

Lemma A.1 *Let X_α, X_β be elements of an arbitrary algebra g in some matrix representation with $2n \times 2n$ matrices. Furthermore, let $\bar{c}_I, c_J$ be creators viz. annihilators operating on some Fock space $\mathcal{F}$ with vacuum $|0\rangle$. Here I, J denote a multi-index (i, σ), where $i = 1, \ldots, n$ is an arbitrary quantum number and $\sigma \in \{B, F\}$ distinguishes between bosonic and fermionic operators.*
The mapping

$$X_\alpha \mapsto Y_\alpha = (c_{(1,B)}, \ldots c_{(n,F)}) X_\alpha \begin{pmatrix} \bar{c}_{(1,B)} \\ \vdots \\ \bar{c}_{(n,F)} \end{pmatrix}$$

is an algebra homomorphism, i.e. preserves the Lie bracket and the vector space structure.

Proof: We calculate the commutator of Y_α an Y_β:

$$\begin{aligned}
[Y_\alpha, Y_\beta] &= [c_I X_{\alpha_{IJ}} \bar{c}_J, c_K X_{\beta_{KL}} \bar{c}_L] \\
&= X_{\alpha_{IJ}} X_{\beta_{KL}} \left(c_I \bar{c}_J c_K \bar{c}_L - c_K \bar{c}_L c_I \bar{c}_J \right) \\
&= X_{\alpha_{IJ}} X_{\beta_{KL}} \Big((-1)^{|J||K|} c_I \bar{c}_L \delta_{JK} - (-1)^{|J||K|} c_I c_K \bar{c}_J \bar{c}_L - \\
&\qquad - \Big((-1)^{|I||L|} c_K \bar{c}_J \delta_{IL} - (-1)^{|I||L|} c_K c_I \bar{c}_L \bar{c}_J \Big) \Big) \\
&= X_{\alpha_{IJ}} X_{\beta_{KL}} \Big((-1)^{|J||K|} c_I \bar{c}_L \delta_{JK} - (-1)^{|I||L|} c_K \bar{c}_J \delta IL \Big) \\
&= c_I X_{\alpha_{IJ}} X_{\beta_{JL}} \bar{c}_L - c_K X_{\beta_{KL}} X_{\alpha_{LJ}} \bar{c}_J \\
&\overset{\text{renaming of var's}}{=} c_I X_{\alpha_{IJ}} X_{\beta_{JL}} \bar{c}_L - c_I X_{\beta_{IJ}} X_{\alpha_{JL}} \bar{c}_L \\
&= c_I X_{\alpha_{IJ}} X_{\beta_{JL}} \bar{c}_L - c_I X_{\beta_{IJ}} X_{\alpha_{JL}} \bar{c}_L \\
&= c_I [X_\alpha, X_\beta]_{IL} \bar{c}_L
\end{aligned}$$

The vector space structure is trivially preserved. □

This lemma can now be used to arrive at a "second quantized" form of the ψ's and $\tilde{\psi}$'s and their products introduced in 2.1. Since we know that $\tilde{\psi}\psi$ is an element of the osp algebra, we simply set

$$\psi = \begin{pmatrix} b_1^1 & \cdots & b_n^1 & \bar{b}_1^1 & \cdots & \bar{b}_n^1 & f_1^1 & \cdots & f_n^1 & \bar{f}_1^1 & \cdots & \bar{f}_n^1 \\ \vdots & & & & & & & & & & & \vdots \\ b_1^N & \cdots & b_n^N & \bar{b}_1^N & \cdots & \bar{b}_n^N & f_1^N & \cdots & f_n^N & \bar{f}_1^N & \cdots & \bar{f}_n^N \end{pmatrix}$$

By (2.4) we get then:

$$\tilde{\psi} = \begin{pmatrix} -\bar{b}_1^1 & \cdots & -\bar{b}_1^N \\ \vdots & & \vdots \\ -\bar{b}_n^1 & \cdots & -\bar{b}_n^N \\ b_1^1 & \cdots & b_1^N \\ \vdots & & \vdots \\ b_n^1 & \cdots & b_n^N \\ \bar{f}_1^1 & \cdots & \bar{f}_1^N \\ \vdots & & \vdots \\ \bar{f}_n^1 & \cdots & \bar{f}_n^N \\ f_1^1 & \cdots & f_1^N \\ \vdots & & \vdots \\ f_n^1 & \cdots & f_n^N \end{pmatrix}$$

Thus we get for $\tilde{\psi}\psi$ ($n=1$, $N=2$):

$$\tilde{\psi}\psi = \begin{pmatrix} -\bar{b}^1 b^1 - \bar{b}^2 b^2 & -\bar{b}^1 \bar{b}^1 - \bar{b}^2 \bar{b}^2 & -\bar{b}^1 f^1 - \bar{b}^2 f^2 & -\bar{b}^1 \bar{f}^1 - \bar{b}^2 \bar{f}^2 \\ b^1 b^1 + b^2 b^2 & b^1 \bar{b}^1 + b^2 \bar{b}^2 & b^1 f^1 + b^2 f^2 & b^1 \bar{f}^1 + b^2 \bar{f}^2 \\ \bar{f}^1 b^1 + \bar{f}^2 b^2 & \bar{f}^1 \bar{b}^1 + \bar{f}^2 \bar{b}^2 & \bar{f}^1 f^1 + \bar{f}^2 f^2 & \bar{f}^1 \bar{f}^1 + \bar{f}^2 \bar{f}^2 \\ f^1 b^1 + f^2 b^2 & f^1 \bar{b}^1 + f^2 \bar{b}^2 & f^1 f^1 + f^2 f^2 & f^1 \bar{f}^1 + f^2 \bar{f}^2 \end{pmatrix}$$

For larger n, N this scheme can be easily extended. This matrix has the structure of an osp(2,2) element when given in matrix representation (cf. [Bere 87]). Each of its entries can be identified with a generator given at the beginning of this appendix[1].

A.2 Pure Generators

In this and the following section we calculate the commutators of our osp generators.

Pure Bosonic Commutators The commutators that contain only bosonic creators and annihilators have to be those of the symplectic group. They are:

$$\begin{aligned} [B^{ab}, B^{cd}] &= 0 \\ [B_{ab}, B_{cd}] &= 0 \\ [B_{ab}, B^{cd}] &= b^i_a b^i_b \bar{b}^i_c \bar{b}^i_d - \bar{b}^i_c \bar{b}^i_d b^i_a b^i_b \qquad \text{The terms with mixed color indices commute trivially} \\ &= b^i_a \bar{b}^i_d \delta_{bc} + b^i_b \bar{b}^i_d \delta_{ac} + \bar{b}^i_c b^i_b \delta_{ad} + \bar{b}^i_c b^i_a \delta_{bd} \\ &= B^d_a \delta_{bc} + B^d_b \delta_{ac} + B^c_a \delta_{ad} + B^c_a \delta_{bd} \\ [B_{ab}, B^c_d] &= B_{ad}\delta_{bc} + B_{bd}\delta_{ac} \\ [B^{ab}, B^c_d] &= -B^{ac}\delta_{bd} - B^{bc}\delta_{ad} \\ [B^a_b, B^c_d] &= B^a_d \delta_{bc} - B^c_b \delta_{ad} \end{aligned}$$

The remaining non-trivial commutators were obtained in the same way as $[B_{ab}, B^{cd}]$. When comparing these results with the sp(2n) commutators (cf. [Pere 86]) one can see the right behavior.

Pure Fermionic Generators To compute the commutators for the FF sector generators (which are obviously *bosonic*) we follow the line of the preceding paragraph:

[1] The elements of form $\bar{c}c$ and $c\bar{c}$ were written as $\frac{1}{2}c\bar{c} \pm \bar{c}c$ before. This corresponds to the addition of some constants, which in turn does not change the commutators and therefore not the algebra either.

$$
\begin{aligned}
[F^{ab}, F^{cd}] &= 0 \\
[F_{ab}, F_{cd}] &= 0 \\
[F_{ab}, F^{cd}] &= f^i_a f^i_b \bar{f}^i_c \bar{f}^i_d - \bar{f}^i_c \bar{f}^i_d f^i_a f^i_b \\
&= f^i_a \bar{f}^i_d \delta_{bc} - f^i_b \bar{f}^i_d \delta_{ac} - \bar{f}^i_c f^i_b + \bar{f}^i_c f^i_a \\
&= F^c_a \delta_{bd} - F^d_a \delta_{bc} - F^c_b \delta_{ad} + F^d_b \delta_{ac} \\
[F_{ab}, F^c_d] &= F_{ad}\delta_{bc} - F_{bd}\delta_{ad} \\
[F^{ab}, F^c_d] &= -F^{ac}\delta_{bd} - F^{bc}\delta_{ad} \\
[F^a_b, F^c_d] &= F^a_d \delta_{bc} - F^c_b \delta_{ad}
\end{aligned}
$$

Comparing these to the so(2n) generators, one identifies the pure fermionic generators as those of the algebra in question.

Mixed Pure Bosonic and Pure Fermionic Commutators All of the nine commutators of form $[B, F]$ vanish, since the constituting creators and annihilators commute.

A.3 Mixed Generators

When dealing with the mixed generators (i.e. those labeled with a 'G') one has to take care of the fact that these generators are of fermionic nature. This implies the use of anti-commutators in the case of "pure mixed" commutators. This can be seen when looking at the definition of the super-commutator [Zirn 96a].

Commutators of Mixed and Pure Generators In this case we still have to use the commutator. This can also be seen from the fact that a fermion and a boson certainly obey the normal commutation relation (In fact *this* is the motivation to define the super-commutator in such a way).

So this yields for the commutators of G^{ab}:

$$
\begin{aligned}
[G^{ab}, B^{cd}] &= 0 \\
[G^{ab}, B^b_a] &= \tfrac{1}{2}\left(\bar{b}^i_a \bar{f}^i_b b^i_c \bar{b}^i_d + \bar{b}^i_a \bar{f}^i_b \bar{b}^i_d b^i_c - b^i_a \bar{b}^i_d \bar{b}^i_a \bar{f}^i_b - \bar{b}^i_d b^i_c \bar{b}^i_a \bar{f}^i_b\right) \\
&= \tfrac{1}{2}\left(\bar{b}^i_a \bar{f}^i_b b^i_c \bar{b}^i_d - \bar{b}^i_d \bar{f}^i_b \delta_{ac} - \bar{b}^i_a \bar{f}^i_b b^i_c \bar{b}^i_d + \bar{b}^i_a \bar{f}^i_b \bar{b}^i_d b^i_c - \bar{b}^i_d \bar{f}^i_b \delta_{ac} - \bar{b}^i_a \bar{f}^i_b \bar{b}^i_d b^i_c\right) \\
&= -\bar{b}^i_d \bar{f}^i_b \delta_{ac} = -G^{db}\delta_{ac} \\
[G^{ab}, B_{cd}] &= \bar{b}^i_a \bar{f}^i_b b^i_c b^i_d - b^i_c b^i_d \bar{b}^i_a \bar{f}^i_b \\
&= \bar{b}^i_a \bar{f}^i_b b^i_c b^i_d - b^i_c \bar{f}^i_b \delta_{ad} - b^i_d \bar{f}^i_b \delta_{ac} - \bar{b}^i_a \bar{f}^i_b b^i_c b^i_d \\
&= -\tilde{G}^b_c \delta_{ad} - \tilde{G}^b_d \delta_{ac} \\
[G^{ab}, F^{ab}] &= 0 \\
[G^{ab}, F^d_c] &= G^{ad}\delta_{bc} \\
[G^{ab}, F_{cd}] &= G^a_d \delta_{bc} - G^a_c \delta_{bd}
\end{aligned}
$$

In the following we will omit the calculations and simply state the results. For G^a_b they are

$$\begin{aligned}
[G^a_b, B^{ab}] &= 0 \\
[G^a_b, B^d_c] &= -G^d_b \delta_{ac} \\
[G^a_b, B_{cd}] &= -G_{cb}\delta_{ad} - G_{db}\delta_{ac} \\
[G^a_b, F^{cd}] &= G^{ad}\delta_{bc} - G^{ac}\delta_{bd} \\
[G^a_b, F^d_c] &= -G^a_c \delta_{bd} \\
[G^a_b, F_{cd}] &= 0
\end{aligned}$$

For $\tilde{G}^a_b$ we get

$$\begin{aligned}
[\tilde{G}^b_a, B^{cd}] &= G^{db}\delta_{ac} + G^{cb}\delta_{ad} \\
[\tilde{G}^b_a, B^d_c] &= \tilde{G}^b_c \delta_{ad} \\
[\tilde{G}^b_a, B_{cd}] &= 0 \\
[\tilde{G}^b_a, F^{cd}] &= 0 \\
[\tilde{G}^b_a, F^d_c] &= \tilde{G}^d_a \delta_{bc} \\
[\tilde{G}^b_a, F_{cd}] &= G_{ad}\delta_{bc} - G_{ac}\delta_{bd}
\end{aligned}$$

And finally for G_{ab}

$$\begin{aligned}
[G_{ab}, B^{cd}] &= G^d_b \delta_{ac} + G^c_b \delta_{ad} \\
[G_{ab}, B^d_c] &= G_{cb}\delta_{ad} \\
[G_{ab}, B_{cd}] &= 0 \\
[G_{ab}, F^{cd}] &= \tilde{G}^d_a \delta_{bc} - \tilde{G}^c_a \delta_{bd} \\
[G_{ab}, F^d_c] &= -G_{ac}\delta_{bd} \\
[G_{ab}, F_{cd}] &= 0
\end{aligned}$$

Commutators of Mixed Generators In this case, where we are calculating the super-commutator of two fermions, we have to use the anti-commutator:

$$\begin{aligned}
\{G_{ab}, G_{ab}\} &= b^i_a f^i_b b^i_c b^i_d + b^i_c f^i_d b^i_a f^i_b \\
&= b^i_a b^i_c (f^i_b f^i_d + f^i_d f^i_b) \\
&= 0 \\
\{G_{ab}, \tilde{G}^d_c\} &= B_{ac}\delta_{bd} \\
\{G_{ab}, G^c_d\} &= F_{bd}\delta_{ac} \\
\{G_{ab}, G^{cd}\} &= F^b_d \delta_{ac} + B^a_c \delta_{bd} \\
\{\tilde{G}^b_a, \tilde{G}^d_c\} &= 0 \\
\{\tilde{G}^b_a, G^c_d\} &= F^b_d \delta_{ac} + B^c_a \delta_{bd} \\
\{\tilde{G}^b_a, G^{cd}\} &= F^{bd}\delta_{ac} \\
\{G^a_b, G^c_d\} &= 0 \\
\{G^a_b, G^{cd}\} &= B^{ac}\delta_{bd} \\
\{G^{ab}, G^{cd}\} &= 0
\end{aligned}$$

Appendix B

The so(N) Generator

B.1 Calculation of $\hat{F}^{ij}$

The so(N) generators are obtained in a similar fashion to the osp generators. While for the osp we had to calculate $\tilde{\psi}\psi$ we will now go for $\psi\tilde{\psi}$ (and again: $n = 1,\ N = 2$):

$$\psi\tilde{\psi} = \begin{pmatrix} -b^1\bar{b}^1 + \bar{b}^1 b^1 + f^1\bar{f}^1 + \bar{f}^1 f^1 & -b^1\bar{b}^2 + \bar{b}^1 b^2 + f^1\bar{f}^2 + \bar{f}^1 f^2 \\ -b^2\bar{b}^1 + \bar{b}^2 b^1 + f^2\bar{f}^1 + \bar{f}^2 f^1 & -b^2\bar{b}^2 + \bar{b}^2 b^2 + f^2\bar{f}^2 + \bar{f}^2 f^2 \end{pmatrix}$$

The elements on the diagonal vanish due to supersymmetry cancellation leaving a skew symmetric matrix as expected. The generator can then be read off (general case) to be:

$$\hat{F}^{ij} = \bar{b}^i_a b^j_a - b^i_a \bar{b}^j_a + \bar{f}^i_a f^j_a + f^i_a \bar{f}^j_a$$

B.2 The Commutator

With $\hat{F}$ as given above we can now calculate the commutator:

$$\begin{aligned} [\hat{F}^{ij}, \hat{F}^{kl}] &= \left(\bar{b}^i_a b^j_a - b^i_a \bar{b}^j_a + \bar{f}^i_a f^j_a + f^i_a \bar{f}^j_a\right)\left(\bar{b}^k_b b^l_b - b^k_b \bar{b}^l_b + \bar{f}^k_b f^l_b + f^k_b \bar{f}^l_b\right) - \\ &\quad -\textit{(vice versa)} \\ &= \bar{b}^i_a b^j_a \bar{b}^k_b b^l_b - \bar{b}^i_a b^j_a b^k_b \bar{b}^l_b - b^i_a \bar{b}^j_a \bar{b}^k_b b^l_b + b^i_a \bar{b}^j_a b^k_b \bar{b}^l_b \\ &\quad + \bar{f}^i_a f^j_a \bar{f}^k_b f^l_b + \bar{f}^i_a f^j_a f^k_b \bar{f}^l_b + f^i_a \bar{f}^j_a \bar{f}^k_b f^l_b + f^i_a \bar{f}^j_a f^k_b \bar{f}^l_b \\ &\quad - \bar{b}^k_b b^l_b \bar{b}^i_a b^j_a + b^k_b \bar{b}^l_b \bar{b}^i_a b^j_a + \bar{b}^k_b b^l_b b^i_a \bar{b}^j_a - b^k_b \bar{b}^l_b b^i_a \bar{b}^j_a \\ &\quad - \bar{f}^k_b f^l_b \bar{f}^i_a f^j_a - f^k_b \bar{f}^l_b \bar{f}^i_a f^j_a - \bar{f}^k_b f^l_b f^i_a \bar{f}^j_a - f^k_b \bar{f}^l_b f^i_a \bar{f}^j_a \end{aligned}$$

$$
\begin{aligned}
&= \delta^{jk}_{ab}\bar{b}^i_a b^l_b - \delta^{li}_{ab}\bar{b}^k_b b^j_a + \delta^{ik}_{ab} b^j_a \bar{b}^l_b - \delta^{jl}_{ab} b^k_b \bar{b}^i_a \\
&\quad + \delta^{jl}_{ab} b^i_a \bar{b}^k_b - \delta^{ik}_{ab} b^l_b \bar{b}^j_a - \delta^{jk}_{ab} b^i_a \bar{b}^l_b + \delta^{il}_{ab} b^k_b \bar{b}^j_a \\
&\quad + \delta^{jk}_{ab} \bar{f}^i_a f^l_b - \delta^{il}_{ab} \bar{f}^k_b f^j_a + \delta^{jl}_{ab} \bar{f}^i_a f^k_b - \delta^{ik}_{ab} \bar{f}^l_b f^j_a \\
&\quad + \delta^{jl}_{ab} f^i_a \bar{f}^k_b - \delta^{ik}_{ab} f^l_b \bar{f}^j_a + \delta^{jk}_{ab} f^i_a \bar{f}^l_b - \delta^{il}_{ab} f^k_b \bar{f}^j_a \\
&= \delta^{jk}_{ab}\left(\bar{b}^i_a b^l_b - b^i_a \bar{b}^l_b + \bar{f}^i_a f^l_b + f^i_a \bar{f}^l_b\right) - \delta^{il}_{ab}\left(\bar{b}^k_b b^j_a - b^k_b \bar{b}^j_a + \bar{f}^k_b f^j_a + f^k_b \bar{f}^j_a\right) \\
&\quad - \delta^{ik}_{ab}\left(\bar{b}^l_b b^j_a - b^l_b \bar{b}^j_a + \bar{f}^l_b f^j_a + f^l_b \bar{f}^j_a\right) + \delta^{jl}_{ab}\left(\bar{b}^i_a b^k_b - b^i_a \bar{b}^k_b + \bar{f}^i_a f^k_b + f^i_a \bar{f}^k_b\right) \\
&= \delta^{jk}\hat{F}^{il} - \delta^{il}\hat{F}^{jk} - \delta^{ik}\hat{F}^{jl} + \delta^{jl}\hat{F}^{ik}
\end{aligned}
$$

From this result we can read off the structure constants of the algebra under investigation. Comparing these to the well known structure constants of the so(N) the identification of the special orthogonal algebra is easy. □

Appendix C

The Flavor Projector – SO(N) Representation

In this short appendix we prove the validity of equation (2.19). Define the matrix X that corresponds to $\hat{F}^{ij}$ by:

$$\mathrm{ad}[\hat{F}^{ij}]\bar{c}^k_A \stackrel{\mathrm{def}}{=} \sum_l (X^{ij})^{kl}\bar{c}^l_A$$

X can be read off from this commutator to be

$$\mathrm{ad}[\hat{F}^{ij}]\bar{c}^k_A = \delta^{jk}\bar{c}^i_A - (-1)^{|A|}\delta^{ki}\bar{c}^j_A.$$

Since the X are so(N) generators we can write for arbitrary $O \in \mathrm{SO(N)}$:

$$\ln O = \sum a_{ij}(O)X^{ij}$$

where $a_{ij}(O)$ are appropiate factors. Taking the $T'_O = \exp\left\{a_{ij}(O)\hat{F}^{ij}\right\}$ we get

$$\begin{aligned} \mathrm{Ad}[T'_O]\bar{c}^l_A &= \exp\left(a_{ij}(O)\mathrm{ad}[\hat{F}^{ij}]\right)\bar{c}^l_A \\ &= \left[\underbrace{\exp\left(a_{ij}(O)X^{ij}\right)}_{=O}\right]^{kl}\bar{c}^l_A \end{aligned}$$

Since the Bose-Fermi states are written in terms of an exponential we have to calculate what happen when we are acting with T'_O on a collection of creators:

$$\begin{aligned} T'_O\bar{c}^i_A\bar{c}^j_A\ldots|0\rangle &= T'_O\bar{c}^i_A T'^{-1}_O T'_O\bar{c}^j_A T'^{-1}_O \ldots \underbrace{T'^{-1}_O|0\rangle}_{=|0\rangle} \\ &= \mathrm{Ad}[T'_O]\bar{c}^i_A\mathrm{Ad}[T'_O]\bar{c}^j_A\ldots|0\rangle \\ &= O^{ii'}\bar{c}^{i'}_A O^{jj'}\bar{c}^{j'}_A\ldots|0\rangle \end{aligned}$$

When we now perform a Taylor expansion on $\exp(O^{ij}\bar{c}^j_A\psi^i_A)$ and apply the above, equation (2.19) follows.

Appendix D

Generalized Gaussian Integrals

In order to get rid of the super-numbers in (3.5) we have to perform the Gaussian integration. Unfortunatly there are terms of type $\bar{\phi}\bar{\phi}$ and $\phi\phi$ in the exponent, such that the usual formula as given in the introduction is not applicable. We have, therefore, to derive another Gaussian identity.

D.1 The Superdeterminant

To prepare ourselves for the next section we prove the following theorem, which tells us how to calculate a general Gaussian integral as described above.

Theorem D.1 *The following identity holds for* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{osp}(2\mathrm{n}, 2\mathrm{n})$*:*

$$\int D(\psi, \bar{\psi}) \exp\left[-\frac{1}{2}\left(\bar{\psi}A\psi + \bar{\psi}B\bar{\psi} - \sigma\psi C\psi - \sigma\psi D\bar{\psi}\right)\right] = \mathrm{SDet}^{-1/2}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where ψ *denotes some super-vector,* A*,* B*,* C *and* D *are super-matrices and* σ *denotes the super-parity.*

Proof: To prove this, first note that the integral above equals

$$\int D(\psi, \bar{\psi}) \exp\left[-\frac{1}{2}\left(\bar{\psi}, -\sigma\psi\right)\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}\right].$$

The diagonalizable matrices $M = T\,\mathrm{diag}(\lambda, -\lambda)\,T^{-1}$ – with λ diagonal – are dense in the set of matrices. Thus we are allowed to continue our results to those not diagonalizable. We take then $\begin{pmatrix} \psi' \\ \bar{\psi}' \end{pmatrix} = T^{-1}\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$. Now we have to check whether the side left of M can be reached by transposition and right multiplication with $\tau = \begin{pmatrix} 0 & -\sigma \\ 1 & 0 \end{pmatrix}$. This τ is an involutive automorphism and it is super-symmetric, i.e. $\tau = -\sigma\tau^T = -\tau^T\sigma$. We see now[1]:

$$\left[T^{-1}\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}\right]^{\mathcal{T}} = (\psi^T, \bar{\psi}^T)T^{-1\,T}\tau = (\bar{\psi}^T, -\sigma\psi^T)\tau^{-1}T^{-1\,T}\tau$$

Thus we have $T = \tau^{-1}T^{-1\,T}\tau$ as a condition on T which means that $T \in$ Osp(2n,2n), as expected (cf. [Zirn 96b]). This is very similar to the "usual" case where we need orthogonal matrices for a similarity transformation.

Our measure is furthermore invariant under Osp rotation and thus we arrive at

$$\int D(\psi', \bar{\psi}') \exp\left[-\frac{1}{2}\left(\bar{\psi}', -\sigma\psi'\right)\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}\begin{pmatrix} \psi' \\ \bar{\psi}' \end{pmatrix}\right].$$

The rest is quite clear, since the above is – by the super-commutation relations – equal to

$$\begin{aligned} &\int D(\psi', \bar{\psi}') \exp\left[-\bar{\psi}'\lambda\psi'\right] \\ &= \frac{\prod_{i=1}^{N} \lambda_{i,F}}{\prod_{i=1}^{N} \lambda_{i,B}} \\ &= \mathrm{SDet}^{-1/2}\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \\ &= \mathrm{SDet}^{-1/2}\begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{aligned}$$

The last equality is clear, since $M = T\mathrm{diag}(\lambda, -\lambda)T^{-1}$. □

D.2 Matrix Elements

Our goal is now to obtain an expression for single elements of the supermatrix M. This is done – in the same fashion as in 1.2.3 – in the following corollary:

[1] $(\bullet)^{\mathcal{T}} = (\bullet)^T\tau$ denotes here a special transposition that allows us to get from $\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$ to $(\bar{\psi}^T, -\sigma\psi^T)$.

Corollary D.1 *An arbitrary matrix element $A_{BB}^{k_1k_2}$ can be expressed in terms of*

$$A_{BB}^{k_1k_2} = \mathrm{SDet}^{1/2}M \int D(\psi,\bar\psi)\psi_B^{k_1}\bar\psi_B^{k_2} \exp\left\{-\frac{1}{2}\left(\bar\psi, -\sigma\psi\right) M \begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}\right\}.$$

where the same notational conventions apply as above and $M = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in$ osp(2n, 2n).

Proof: Let's perform the transformation $\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix} \rightarrow \begin{pmatrix}\psi\\ \bar\psi\end{pmatrix} + M^{-1}\begin{pmatrix} J \\ -\sigma\bar J\end{pmatrix}$. Now we have to observe that for the Osp(2n|2n) the following equation holds (cf. [Zirn 96b][2]):

$$\begin{aligned} \forall g \in \mathrm{Osp(2n|2n)}: \quad & g = \tau g^{-1T}\tau^{-1} \\ \Rightarrow \quad & \tau g^{T} = g^{-1}\tau \\ \text{Differentiation yields:} \quad & \tau X^{T} = -X\tau \\ \Rightarrow \quad & X^{-1T} = -\tau^{-1}X^{-1}\tau \end{aligned}$$

Here we assumed that the algebra element $X \in$ osp(2n,2n) has an inverse, but this is quite reasonable, since otherwise the SDet^{-1} expression would make no sense.

Turning back to the expression above we get:

$$\begin{aligned} \left[\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix} + M^{-1}\begin{pmatrix} J \\ -\sigma\bar J\end{pmatrix}\right]^{\mathcal{T}} &= \left[\left(\psi^T, \bar\psi^T\right) + \left(J^T, -\sigma\bar J^T\right) M^{-1T}\right]\tau \\ &= \left[\left(\psi^T, \bar\psi^T\right) - \left(J^T, -\sigma\bar J^T\right)\tau^{-1}M^{-1}\tau\right]\tau \\ &= \left(\psi^T, \bar\psi^T\right)\tau + \left(J^T, -\sigma\bar J^T\right)\tau^{-1}M^{-1}\tau\tau \\ &= \left(\bar\psi^T, -\sigma\psi^T\right) + \left(\bar J^T, J^T\right)\tau^{T-1}\tau^{-1}M^{-1}\tau\tau \\ &= \left(\bar\psi^T, -\sigma\psi^T\right) + \left(\bar J^T, J^T\right)M^{-1} \end{aligned}$$

Therefore we can finally write:

$$\begin{aligned} & \int D(\psi,\bar\psi)\psi_B^{k_1}\bar\psi_B^{k_2} \exp\left\{-\frac{1}{2}\left(\bar\psi, -\sigma\psi\right) M \begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}\right\} \\ = \; & \left.\frac{\partial^2}{\partial J_B^{k_1}\partial \bar J_B^{k_2}}\right|_{J,\bar J=0} \int \exp\left\{-\frac{1}{2}\left[(\bar\psi, -\sigma\psi) + (\bar J, J)M^{-1}\right] M \left[\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix} + M^{-1}\begin{pmatrix} J \\ -\sigma\bar J\end{pmatrix}\right] + \right. \end{aligned}$$

[2]The different sign in the super-symmetry of τ stems from the fact that – for our Osp – the fermionic and bosonic sectors are exchanged (cf. eqn. (2.5).

$$
\begin{aligned}
&\quad + \frac{1}{2}(\bar{J}, J) M^{-1} \begin{pmatrix} J \\ -\sigma \bar{J} \end{pmatrix} \Bigg\} \\
&= \left.\frac{\partial^2}{\partial J_B^{k_1} \partial \bar{J}_B^{k_2}}\right|_{J,\bar{J}=0} \mathrm{SDet}^{-1/2} M \exp \left\{ \frac{1}{2}(\bar{J}, J) M^{-1} \begin{pmatrix} J \\ -\sigma \bar{J} \end{pmatrix} \right\} \\
&= \mathrm{SDet}^{-1/2} M \frac{1}{2} \left[A_{BB}^{k_1 k_2} - D_{BB}^{k_2 k_1} \right] \\
&= \mathrm{SDet}^{-1/2} M \frac{1}{2} \left[A_{BB}^{k_1 k_2} + (A^T)_{BB}^{k_2 k_1} \right] \\
&= A_{BB}^{k_1 k_2} \mathrm{SDet}^{-1/2} M
\end{aligned}
$$

This proves the corollary. □

Appendix E

Disorder Average for $p = 1$

We calculated in 3.2.2 the lattice action for the sigma model corresponding to $p = \frac{1}{2}$ in the probability distribution. Therefore we had to integrate over the full O(1). In order to arrive at a representation in terms of Z-matrices we have to integrate over the SO(1) since this corresponds to just one point – $p = 1$. The general principles as in 3.2.2 still apply, but this time things will get a little bit more complicated.

We start with equation (3.4) in this case, too. When we then perform the "full" color-flavor transformation for the SO(1) – including the baryonic term – we arrive at something like:

$$\langle g_{MN} \rangle^{p=1} = \langle g_{MN} \rangle^{p=1/2} + \langle g_{MN} \rangle^{B}.$$

Here we have introduced the baryonic term $\langle \bullet \rangle^B$. Since this is the only unknown expression up to now we will restrict ourselves to the calculation of this term. Furthermore, since most of this term is very similar to equation (3.5), we will look at a single $Z, \tilde{Z}$ term in the exponent only (i.e. for *one* node). This looks after C-F transformation such as:

$$\begin{aligned}
J \sum_{T,T',\tau=\pm} \tanh\theta_1 \Big[& \bar{\phi}_{\tau\hat{T}}\left(P(00)\right) Z^{P(00)}_{\hat{T}\hat{T}'} \bar{\phi}_{\tau'\hat{T}'}\left(P(00)\right) \\
& + \phi_{\tau F}\left(P(00); e\right) Z^{P(00)}_{Fe\hat{T}'} \bar{\phi}_{\tau'\hat{T}'}\left(P(00)\right) \\
& + \bar{\phi}_{\tau\hat{T}}\left(P(00)\right) Z^{P(00)}_{\hat{T}Fe'} \phi_{\tau' F}\left(P(00); e'\right) \\
& + \phi_{\tau\hat{T}}\left(P - (t-1)e_x(10)\right) \tilde{Z}^{P(00)}_{\hat{T}\hat{T}'} \phi_{\tau'\hat{T}'}\left(P - (t'-1)e_x(10)\right) \\
& + \bar{\phi}_{\tau F}\left(P - (e-1)e_x(10); e\right) \tilde{Z}^{P(00)}_{Fe\hat{T}'} \phi_{\tau'\hat{T}'}\left(P - (t'-1)e_x(10)\right) \\
& + \phi_{\tau\hat{T}}\left(P - (t-1)e_x(10)\right) \tilde{Z}^{P(00)}_{\hat{T}Fe'} \bar{\phi}_{\tau' F}\left(P - (e'-1)e_x(10); e'\right) \Big]
\end{aligned}$$

Here we have introduced $T = (\sigma, t)$ as a shorthand notation. The hatted indices $\hat{T}$ correspond to the whole range except, again, (F, e). To obtain the full expression for $\langle \bullet \rangle^B$ we have to apply this transformation to the other nodes as well.

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Köln, September 1997

Gerald Beuchelt

Declaration – Erklärung

I herby declare that I made this thesis myself. I did not use any other sources and auxiliary means than those specified. Citations are marked as such.

Ich erkläre hiermit, daß ich meine Diplomarbeit selbständig angefertigt, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und Zitate kenntlich gemacht habe.

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